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# Twisted Superspace for N=D=2 Super BF and Yang-Mills with Dirac-Kähler Fermion Mechanism

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**Abstract**

We propose a twisted D=N=2 superspace formalism. The relation between the twisted super charges including the BRST charge, vector and pseudo scalar super charges and the N=2 spinor super charges is established. We claim that this relation is essentially related with the Dirac-Kähler fermion mechanism. We show that a fermionic bilinear form of twisted N=2 chiral and anti-chiral superfields is equivalent to the quantized version of BF theory with the Landau type gauge fixing while a bosonic bilinear form leads to the N=2 Wess-Zumino action. We then construct a Yang-Mills action described by the twisted N=2 chiral and vector superfields, and show that the action is equivalent to the twisted version of the D=N=2 super Yang-Mills action, previously obtained from the quantized generalized topological Yang-Mills action with instanton gauge fixing.

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# 1 Introduction

It is obviously one of the most fundamental issues to understand the origin of super symmetry if any. The topological field theory proposed by Witten[1] posed a possibility that N=2 Yang-Mills action can be generated by the twisted version of super Yang-Mills action which is equivalent to the quantized topological Yang-Mills theory. It was soon recognized that Witten's formulation could be derived from the "partially" BRST gauge fixed action of the topological Yang-Mills action with instanton gauge-fixing [2-5]. Here the twisting procedure played an important role to generate matter fermions from the ghost related fermions. After this proposal there have been many systematic investigations on this subject[6-12], discovery of a new type of supersymmetry[13-27], related topological actions[28-32] and a new type of topological actions[33, 34].

In the investigations of the topological field theories of Schwarz type; Chern-Simons action and BF actions, a new type of vector supersymmetry was discovered[13-16]. It was recognized that this vector supersymmetry belongs to a twisted version of an extended supersymmetry of N=2 or N=4. The origin of the vector supersymmetry was recognized in some particular examples to be related to the fact that the energy-momentum tensor can be expressed as a pure BRST variation[35-37]. It was later stressed that a (pseudo) scalar supersymmetry was also accompanied together with BRST and vector supersymmetry[38]. The connection of the extended supersymmetry and the quantization procedure of anti-field formalism by Batalin and Vilkovisky was also investigated[39, 40].

Even with those intensive investigations on the relation between the quantized topological field theories and twisted supersymmetry, there still remained unclear the fundamental understandings of the relations between the BRST symmetry and the vector and (pseudo) scalar supersymmetry, quantization of topological field theories, and the twisting mechanism.

In the previous paper[12] one of the authors and Tsukioka showed that the two dimensional twisted N=2 super Yang-Mills action can be derived by instanton gauge fixing of the two dimensional version of the topological Yang-Mills action of the generalized gauge theory. This quantized action has the close connection with N=2 super Yang-Mills action obtained from the different context[41, 42]. This two dimensional formulation is completely parallel to the Witten's four dimensional version of the topological field theory. In the paper it was explicitly shown how the BRST charge, the charges of the vector supersymmetry and the pseudo scalar supersymmetry constitute twisted N=2 super charges and are related to the spinor super charges of N=2 supersymmetry in two dimensions. The twisting procedure generating the matter fermions from ghost related fields is essentially equivalent to the Dirac-Kähler fermion formulation[12]. The R-symmetry of N=2 super symmetry is nothing but the "flavor" symmetry of the Dirac-Kähler fermion fields.

The mysterious relations between the super charges of N=2 supersymmetry and BRST charges and the newly discovered vector and peudo schalar supersymmetry charges are cleared up. They have the same relations as the Dirac-Kähler

fermion fields and anti-symmetric tensor fields of differential forms in the Dirac-Kähler fermion formulation. The mechanism how the matter fermions are generated from the ghost related fields by the twisting procedure is essentially Dirac-Kähler fermion mechanism itself, *i.e.* the matter fermions are generated by the ghost related fields which have anti-symmetric tensor suffixes.

In this article we show that there exists twisted superspace formalism which naturally accommodates the twisting procedure and the Dirac-Kähler fermion mechanism. So far most of the examples of the twisted supersymmetric models have only on-shell N=2 or N=4 supersymmetry. We show that the quantized BF and super Yang-Mills actions can have off-shell N=2 supersymmetry by introducing auxiliary fields. We show that those quantized BF and Yang-Mills actions can be equivalently formulated by a simple form of twisted N=2 chiral super fields and vector superfield. We can thus establish the twisted superspace formalism in two dimensions which we claim as the most essential formulation behind the twisted supersymmetry.

In this article Dirac-Kähler fermion formulation plays a fundamental role. The original idea that fermion field can be formulated by differential forms is old back to Ivanenko and Landau[43]. It was later shown by Kähler [44] that Dirac equation is constructed from the direct sum of inhomogeneous differential forms which is called Dirac-Kähler field[45-50]. Here all the degrees of differential forms are needed to express the Dirac-Kähler fields with "flavor" suffix which is shown to denote the extended supersymmetry suffix as well.

In formulating the two dimensional super Yang-Mills action, we needed to start from the two dimensional version of topological Yang-Mills action of the generalized gauge theory. The generalized gauge theory is the generalization of the standard gauge theory by introducing all the degrees of differential forms and can be defined in arbitrary dimensions. In particular the generalization of the Chern-Simons actions into arbitrary dimensions was proposed by one of the authors (N.K.) and Watabiki[33, 34]. It is interesting to note that the generalized gauge theory also introduce all the degrees of differential forms as gauge fields and parameters together with quaternion structure.

This paper is organized as follows. In section 2 we show that the quantized BF models with auxiliary fields in two dimensions have off-shell N=2 twisted super symmetry. In section 3 we quantize the generalized two-dimensional topological Yang-Mills theory with the instanton gauge fixing and show that the quantized action leads to twisted N=2 super Yang-Mills action at the on-shell level. In section 4 we formulate the twisted N=2 super space formalism for chiral super fields and vector super fields and show that the quantized BF actions and the quantized topological Yang-Mills action of generalized gauge theory in two dimensions can be written down by chiral and vector superfields and thus possess off-shell N=2 twisted super symmetry. In section 5 we explain that the twisting mechanism is essentially equivalent to the Dirac-Kähler fermion formulation. Conclusions and discussions are given in the final section.

# 2 A simple model of twisted N=D=2

## 2.1 Twisted D=N=2 Supersymmetry Algebra

In this subsection, we summarize the construction of the twisted D=N=2 supersymmetry algebra[3, 4]. Throughout this paper, we consider the two-dimensional Euclidean space-time. The notation is summarized in Appendix.

We first introduce D=N=2 supersymmetry algebra without a central extension:

$$\begin{aligned}
\{Q_{\alpha i}, Q_{\beta j}\} &= 2\delta_{ij}\gamma^{\mu}{}_{\alpha\beta}P_{\mu}, \\
[P_{\mu}, Q_{\alpha i}] &= 0, \\
[J, Q_{\alpha i}] &= \tfrac{i}{2}(\gamma^5)_{\alpha}{}^{\beta}Q_{\beta i}, \\
[R, Q_{\alpha i}] &= \tfrac{i}{2}(\gamma^5)_{i}{}^{j}Q_{\alpha j}, \\
[J, P_{\mu}] &= i\epsilon_{\mu}{}^{\nu}P_{\nu}, \\
[P_{\mu}, P_{\nu}] &= [P_{\mu}, R] = [J, R] = 0.
\end{aligned} \tag{2.1}$$

Here $Q_{\alpha i}$ is super-charge, where the left-indices $\alpha(=1,2)$ and the right-indices $i(=1,2)$ are Lorentz spinor and internal spinor suffixes labeling two different N=2 supercharges, respectively. We can take these operators to be Majorana. $P_{\mu}$ is generator of translation. $J$ and $R$ are generators of $SO(2)$ Lorentz and $SO(2)_I$ internal rotation called $R$ symmetry, respectively.

The essential meaning of the topological twist is to identify the isospinor indices as the spinor ones. Then isospinor should transform as spinor under the Lorentz transformation. This will lead to a redefinition of the energy-momentum tensor and the Lorentz rotation generator.

We can redefine the energy-momentum tensor $T_{\mu\nu}$ as the following relation without breaking the conservation law:

$$T'_{\mu\nu} = T_{\mu\nu} + \epsilon_{\mu\rho}\partial^{\rho}R_{\nu} + \epsilon_{\nu\rho}\partial^{\rho}R_{\mu}, \tag{2.2}$$

where $R_{\mu}$ is the conserved current associated with $R$ symmetry[1, 10, 18]. This modification leads to a redefinition of the Lorentz generator,

$$J' = J + R. \tag{2.3}$$

This rotation group is interpreted as the diagonal subgroup of $SO(2) \times SO(2)_I$.

Now the super charges have double spinor indices and thus can be decomposed into the following scalar, vector and pseudo-scalar components:

$$\mathbf{Q}_{\alpha\beta} = \left(\mathbf{1}Q + \gamma^{\mu}Q_{\mu} + \gamma^5\tilde{Q}\right)_{\alpha\beta}, \tag{2.4}$$

or equivalently,

$$
\begin{aligned}
Q &= \frac{1}{2}\mathrm{Tr}(\mathbf{Q}) \\
Q_\mu &= \frac{1}{2}\mathrm{Tr}(\gamma_\mu \mathbf{Q}) \\
\tilde{Q} &= -\frac{1}{2}\mathrm{Tr}(\gamma^5 \mathbf{Q})
\end{aligned}
\tag{2.5}
$$

The relations (2.1) can be rewritten by the twisted generators:

$$
\begin{aligned}
&\{Q, Q_\mu\} = P_\mu,\ \{\tilde{Q}, Q_\mu\} = -\epsilon_{\mu\nu}P^\nu,\ Q^2 = \tilde{Q}^2 = \{Q, \tilde{Q}\} = \{Q_\mu, Q_\nu\} = 0, \\
&[Q, P_\mu] = [\tilde{Q}, P_\mu] = 0,\ [Q_\mu, P_\nu] = 0, \\
&[J', Q] = [J', \tilde{Q}] = 0,\ [J', Q_\mu] = i\epsilon_{\mu\nu}Q^\nu, \\
&[R, Q] = \tfrac{i}{2}\tilde{Q},\ [R, Q_\mu] = \tfrac{i}{2}\epsilon_{\mu\nu}Q^\nu,\ [R, \tilde{Q}] = -\tfrac{i}{2}Q, \\
&[J', P_\mu] = i\epsilon_{\mu\nu}P^\nu, \\
&[P_\mu, P_\nu] = [P_\mu, R] = [J', R] = 0.
\end{aligned}
\tag{2.6}
$$

This is the twisted D=N=2 supersymmetry algebra.

## 2.2 N=D=2 Super BF from quantized BF theory

In order to reveal the fundamental relation between the quantization of topological action and supersymmetry, we consider the simplest example of the two-dimensional abelian BF theory[28, 29]. The action is given by

$$
S_{ABF} = \int_{M_2} d^2x \epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu \tag{2.7}
$$

where $M_2$ is a two-dimensional Euclidean manifold, $\epsilon^{12} = \epsilon_{12} = 1$, and $\phi$ is a 0-form field. This action has the following obvious gauge invariance:

$$
\begin{aligned}
\delta\phi &= 0, \\
\delta\omega_\mu &= \partial_\mu v.
\end{aligned}
\tag{2.8}
$$

To obtain the gauge fixed action, we introduce the ghost field $c$, anti-ghost field $\bar{c}$, and the auxiliary field $b$, and define their BRST transformation as follows:

$$
\begin{aligned}
s\phi &= 0, \\
s\omega_\mu &= \partial_\mu c, \\
sc &= 0, \\
s\bar{c} &= -ib, \\
sb &= 0,
\end{aligned}
\tag{2.9}
$$

where $s^2 = 0$. Then we can obtain the gauge fixed action in the Landau gauge,

$$
\begin{aligned}
S_{\text{on-shell AQBF}} &= \int_{M_2} d^2x[\epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu + is(\bar{c}\partial^\mu\omega_\mu)], \\
&= \int_{M_2} d^2x[\epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu + b\partial^\mu\omega_\mu - i\bar{c}\partial^\mu\partial_\mu c].
\end{aligned}
\tag{2.10}
$$

This action has not only the BRST symmetry (2.9), but also has two more fermionic symmetries[13-27] as shown in the following table 1. We can see that these operators

| $\phi^A$ | $s\phi^A$ | $s_\mu\phi^A$ | $\widetilde{s}\phi^A$ |
|---|---|---|---|
| $\phi$ | $0$ | $-\epsilon_{\mu\nu}\partial^\nu\bar{c}$ | $0$ |
| $\omega_\nu$ | $\partial_\nu c$ | $0$ | $-\epsilon_{\nu\rho}\partial^\rho c$ |
| $c$ | $0$ | $-i\omega_\mu$ | $0$ |
| $\bar{c}$ | $-ib$ | $0$ | $-i\phi$ |
| $b$ | $0$ | $\partial_\mu\bar{c}$ | $0$ |

Table 1: On-shell N=2 twisted super transformation of abelian BF model.

satisfy the following relations if the equations of motion holds:

$$
\begin{aligned}
s^2 = \{s, \tilde{s}\} = \tilde{s}^2 = \{s_\mu, s_\nu\} = 0, \\
\{s, s_\mu\} = -i\partial_\mu, \{\tilde{s}, s_\mu\} = i\epsilon_{\mu\nu}\partial^\nu.
\end{aligned}
\tag{2.11}
$$

This is the twisted D=N=2 supersymmetry algebra which can be recognized by defining the super charge operators,

$$
Q_{\alpha i} = \left(\mathbf{1}s + \gamma^\mu s_\mu + \gamma^5\tilde{s}\right)_{\alpha i}, \tag{2.12}
$$

which satisfy the following extended N=2 supersymmetry:

$$
\{Q_{\alpha i}, Q_{\beta j}\} = 2\delta_{ij}{\gamma^\mu}_{\alpha\beta}P_\mu. \tag{2.13}
$$

Thus the quantuzed BF theory fixed in Landau gauge has the twisted D=N=2 supersymmetry at the on-shell level.

By adding auxilialy fields to eq.(2.10) we can find off-shell N=2 supersymmetric action

$$
S_{\text{off-shell AQBF}} = \int_{M_2} d^2x[\epsilon^{\mu\nu}\phi\partial_\mu\omega_\nu + b\partial^\mu\omega_\mu - i\bar{c}\partial^\mu\partial_\mu c - i\lambda\rho], \tag{2.14}
$$

which has the off-shell extended N=2 supersymmetry invariance shown in Table 2.

It turns out that the off-shell N=2 supersymmetric action is equivalent to the following simple form:

$$
S_{\text{off-shell AQBF}} = \int_{M_2} d^2x s\tilde{s}\frac{1}{2}\epsilon^{\mu\nu}s_\mu s_\nu(-i\bar{c}c). \tag{2.15}
$$

| $\phi^A$ | $s\phi^A$ | $s_\mu\phi^A$ | $\widetilde{s}\phi^A$ |
|---|---|---|---|
| $\phi$ | $i\rho$ | $-\epsilon_{\mu\nu}\partial^\nu\bar{c}$ | $0$ |
| $\omega_\nu$ | $\partial_\nu c$ | $-i\epsilon_{\mu\nu}\lambda$ | $-\epsilon_{\nu\rho}\partial^\rho c$ |
| $c$ | $0$ | $-i\omega_\mu$ | $0$ |
| $\bar{c}$ | $-ib$ | $0$ | $-i\phi$ |
| $b$ | $0$ | $\partial_\mu\bar{c}$ | $-i\rho$ |
| $\lambda$ | $\epsilon^{\mu\nu}\partial_\mu\omega_\nu$ | $0$ | $-\partial_\mu\omega^\mu$ |
| $\rho$ | $0$ | $-\partial_\mu\phi-\epsilon_{\mu\nu}\partial^\nu b$ | $0$ |

Table 2: Off-shell N=2 twisted super transformation of abelian BF model.

This form of action suggests that there is a superspace formulation of extended twisted N=2 supersymmetry.

The extension from abelian group to non-abelian group is straightforward. The off-shell N=2 supersymmetric quantized BF action with non-abelian gauge group is given by:

$$S_{\text{off-shell NABQBF}} = \int_{M_2} d^2x \ \mathrm{Tr}\ [\phi F + b\partial^\mu\omega_\mu + i\partial^\mu\bar{c}D_\mu c - i\lambda\rho], \tag{2.16}$$

where the fields are all Lie algebra valued and

$$F = \epsilon^{\mu\nu}(\partial_\mu\omega_\nu + \omega_\mu\omega_\nu) \tag{2.17}$$

$$D_\mu = \partial_\mu c + [\omega_\mu, c]. \tag{2.18}$$

The off-shell extended N=2 supersymmetry transformation is given in Table 3.

| $\phi^A$ | $s\phi^A$ | $s_\mu\phi^A$ | $\widetilde{s}\phi^A$ |
|---|---|---|---|
| $\phi$ | $[\phi,c]+i\rho$ | $-\epsilon_{\mu\nu}\partial^\nu\bar{c}$ | $0$ |
| $\omega_\nu$ | $D_\nu c$ | $-i\epsilon_{\mu\nu}\lambda$ | $-\epsilon_{\nu\rho}\partial^\rho c$ |
| $c$ | $-c^2$ | $-i\omega_\mu$ | $0$ |
| $\bar{c}$ | $-ib$ | $0$ | $-i\phi$ |
| $b$ | $0$ | $\partial_\mu\bar{c}$ | $-[\phi,c]-i\rho$ |
| $\lambda$ | $F-\{\lambda,c\}$ | $0$ | $-\partial_\mu\omega^\mu$ |
| $\rho$ | $-\{\rho,c\}$ | $-D_\mu\phi-\epsilon_{\mu\nu}\partial^\nu b-i\epsilon_{\mu\nu}\{\partial^\nu\bar{c},c\}$ | $0$ |

Table 3: Off-shell N=2 twisted super transformation of non-abelian BF model.

Just like in the abelian case this off-shell N=2 super symmetric action can be equivalently written down by the same simple form as the abelian case

$$S_{\text{off-shell NABQBF}} = \int_{M_2} d^2x s\tilde{s}\frac{1}{2}\epsilon^{\mu\nu}s_\mu s_\nu \mathrm{Tr}(-i\bar{c}c), \tag{2.19}$$

which again suggests the N=2 twisted superspace formalism.

# 3 N=D=2 super Yang-Mills from quantized generalized topological Yang-Mills

In this subsection, we briefly summarize the formulation of the generalized gauge theory introduced by Kawamoto and Watabiki[33, 34], and the twisted D=N=2 generalized topological Yang-Mills model[12].

## 3.1 Generalized gauge theory

The generalized gauge field $\mathcal{A}$ and the generalized gauge parameter $\mathcal{V}$ are defined as follows:

$$\mathcal{A} = \mathbf{1}\psi + \mathbf{i}\hat{\psi} + \mathbf{j}A + \mathbf{k}\hat{A}, \tag{3.1}$$

$$\mathcal{V} = \mathbf{1}\hat{a} + \mathbf{i}a + \mathbf{j}\hat{\alpha} + \mathbf{k}\alpha. \tag{3.2}$$

Here $(\psi, \alpha)$, $(\hat{\psi}, \hat{\alpha})$, $(A, a)$, and $(\hat{A}, \hat{a})$ are the direct sums of fermionic odd-forms, fermionic even-forms, bosonic odd-forms, and bosonic even-forms, respectively, and they are defined as the following graded Lie algebra valued fields:

$$A = A^a T_a,\ \hat{A} = \hat{A}^\alpha \Sigma_\alpha,\ \psi = \psi^\alpha \Sigma_\alpha,\ \hat{\psi} = \hat{\psi}^a T_a, \tag{3.3}$$

$$\hat{a} = \hat{a}^a T_a,\ a = a^\alpha \Sigma_\alpha,\ \hat{\alpha} = \hat{\alpha}^\alpha \Sigma_\alpha,\ \alpha = \alpha^a T_a, \tag{3.4}$$

where the generators $T_a$ and $\Sigma_\alpha$ satisfy the relations

$$[T_a, T_b] = f_{ab}^c T_c\ ,\ [T_a, \Sigma_\beta] = g_{a\beta}^\gamma \Sigma_\gamma\ ,\ \{\Sigma_\alpha, \Sigma_\beta\} = h_{\alpha\beta}^c T_c. \tag{3.5}$$

The symbols $\mathbf{1}$, $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfy the following quaternion algebra:

$$\mathbf{1}^2 = \mathbf{1}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}, \tag{3.6}$$

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k}, \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \tag{3.7}$$

The generalized Chern-Simons action on the even and odd dimensional manifold are given

$$S_{even} = \int_{M_{even}} \mathrm{Tr}_{\mathbf{k}} \left( \mathcal{AQA} + \frac{2}{3}\mathcal{A}^3 \right), \tag{3.8}$$

$$S_{odd} = \int_{M_{odd}} \mathrm{Str}_{\mathbf{j}} \left( \mathcal{AQA} + \frac{2}{3}\mathcal{A}^3 \right), \tag{3.9}$$

where the traces $\mathrm{Tr}_{\mathbf{k}}(\cdots)$ and $\mathrm{Str}_{\mathbf{k}}(\cdots)$ are defined so as to pick up the coefficient of $\mathbf{k}$ and $\mathbf{j}$ from $(\cdots)$, respectively, and fulfill the following characteristic of traces:

$$\begin{aligned} \mathrm{Tr}[T_a, \cdots] &= \mathrm{Tr}[\Sigma_\alpha, \cdots] = 0, \\ \mathrm{Str}[T_a, \cdots] &= \mathrm{Str}\{\Sigma_\alpha, \cdots\} = 0, \end{aligned} \tag{3.10}$$

where the number of $\Sigma_\alpha$s in $\cdots$ of $\{\Sigma_\alpha, \cdots\}$ should be odd. The differential operator $\mathcal{Q}$ is defined as follows:

$$\mathcal{Q} = \mathbf{j}d. \tag{3.11}$$

where $d = dx^\mu \partial_\mu$.

The generalized Chern-Simons actions (3.8) and (3.9) are invariant under the following gauge transformation:

$$\delta\mathcal{A} = [\mathcal{Q} + \mathcal{A}, \mathcal{V}]. \tag{3.12}$$

This symmetry is larger than the usual gauge symmetry since the generalized gauge parameter $\mathcal{V}$ contains the various form parameters.

As in the usual gauge theory, it is possible to define the generalized Chern character:

$$\mathrm{Str}_{\mathbf{1}}(\mathcal{F}^n) = \mathrm{Str}_{\mathbf{1}}(\mathcal{Q}\Omega_{2n-1}), \tag{3.13}$$

$$\mathrm{Tr}_{\mathbf{i}}(\mathcal{F}^n) = \mathrm{Tr}_{\mathbf{i}}(\mathcal{Q}\Omega_{2n-1}), \tag{3.14}$$

where $\mathcal{F}$ is the generalized curvature

$$\mathcal{F} = \mathcal{Q}\mathcal{A} + \mathcal{A}^2, \tag{3.15}$$

and $\Omega_{2n-1}$ is the generalized Chern-Simons form. Equations (3.13) and (3.14) are bosonic even form and bosonic odd form, respectively. In case of $n = 2$, the topological Yang-Mills type action can be obtained from the generalized Chern-Simons Lagrangian in an even-dimensional manifold $M$ as follows:

$$S_{TYM} = \int_M \mathrm{Str}_{\mathbf{1}}(\mathcal{F}^2) = \int_M \mathrm{Str}_{\mathbf{1}} \left[ \mathcal{Q} \left( \mathcal{A}\mathcal{Q}\mathcal{A} + \frac{2}{3}\mathcal{A}^3 \right) \right]. \tag{3.16}$$

## 3.2 Quantization of generalized topological Yang-Mills action in D=2

The two dimensional generalized gauge field without fermionic components can be taken as the form

$$\mathcal{A}_0 = \mathbf{j}\omega^a T^a + \mathbf{k}(\phi^\alpha + B^\alpha)\Sigma^\alpha, \tag{3.17}$$

where $\phi$, $\omega$, and $B$ are graded Lie algebra valued bosonic 0-, 1-, and 2-form field, respectively. We can take the following algebra as the graded Lie algebra:

$$\{T^a\} = \{1, \gamma^5\}, \{\Sigma^\alpha\} = \{\gamma^1, \gamma^2\}, \tag{3.18}$$

then Str can be taken

$$\mathrm{Str}(\cdots) = \mathrm{Tr}(\gamma^5 \cdots). \tag{3.19}$$

The generalized curvature is given by

$$\begin{aligned}\mathcal{F}_0 &= \mathcal{Q}\mathcal{A}_0 + {\mathcal{A}_0}^2 \\ &= -\mathbf{1}(d\omega + \phi^2 + \{\phi, B\}) + \mathbf{i}(d\phi + [\omega, \phi]).\end{aligned} \tag{3.20}$$

Now two-dimensional topological Yang-Mills action is given as follows:

$$\begin{aligned}S_{TYM} &= \frac{1}{2}\int \mathrm{Str}_{\mathbf{1}} {\mathcal{F}_0}^2 = \frac{1}{2}\int \mathrm{Str}_{\mathbf{1}} \left[\mathcal{Q}\left(\mathcal{A}_0\mathcal{Q}\mathcal{A}_0 + \frac{2}{3}\mathcal{A}_0^3\right)\right] \\ &= \int d^2x \epsilon^{\mu\nu}(F_{\mu\nu}|\phi|^2 + \epsilon^{ab}(D_\mu\phi)_a(D_\nu\phi)_b) \\ &= \int d^2x \epsilon^{\mu\nu}\partial_\mu(2\omega_\nu|\phi|^2 + \epsilon^{ab}\phi_a\partial_\nu\phi_b),\end{aligned} \tag{3.21}$$

where $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, $|\phi|^2 = \phi^a\phi_a$, and $(D_\mu\phi)_a = \partial_\mu\phi_a - 2\epsilon_{ab}\omega_\mu\phi^b$. In this action, the scalar component of even form generators for the 1-form field and odd form generator components of the 2-form field drop out due to the reducible structures of the gauge transformations. Then the generalized gauge transformation turns out to be the following $SO(2)$ gauge transformation:

$$\begin{aligned}\delta_{gauge}\phi_a &= 2v\epsilon_{ab}\phi^b, \\ \delta_{gauge}\omega_\mu &= \partial_\mu v,\end{aligned} \tag{3.22}$$

where $v$ is the 0-form gauge parameter.

The Topological Yang-Mills action (3.21) has the topological shift symmetry:

$$\begin{aligned}\delta_{shift}\phi_a &= u_a, \\ \delta_{shift}\omega_\mu &= u_\mu,\end{aligned} \tag{3.23}$$

where $u_a$ and $u_\mu$ are the shift parameters.

We can see that the action (3.21) is invariant under the following BRST transformation:

$$\begin{aligned}s\phi_a &= 2\epsilon_{ab}\phi^b C - \tilde{C}_a, \\ s\omega_\mu &= \partial_\mu C + \tilde{C}_\mu, \\ sC &= -\eta, \\ s\tilde{C}_a &= 2\epsilon_{ab}(C\tilde{C}^b - \phi^b\eta), \\ s\tilde{C}_\mu &= \partial_\mu\eta, \\ s\eta &= 0,\end{aligned} \tag{3.24}$$

where $C$, $(\tilde{C}_a, \tilde{C}_\mu)$, and $\eta$ are the ghost associated with the $SO(2)$ gauge parameter, and the ghosts of the topological shift symmetry, and the ghost for the ghost of the reducible gauge symmetry, respectively.

We can find a two-dimensional instanton relation of the generalized gauge system by imposing the self- (anti-self-) dual condition:

$$*\mathcal{F}_0 = \pm \mathcal{F}_0. \tag{3.25}$$

Here the dual operator $*$ operates on the differential forms as the Hodge dual operation, furthermore the dual of the generators and the quaternions are defined as follows:

$$*1 = -\gamma^5, \ *\gamma^a = \epsilon^{ab}\gamma_b, \ *\gamma^5 = -1, \tag{3.26}$$

$$*\mathbf{1} = \mathbf{1}, \ *\mathbf{i} = -\mathbf{i}. \tag{3.27}$$

Then we can find the following minimal condition of the action leading to the instanton relations:

$$\begin{aligned} &\frac{1}{2}\int \mathrm{Str}_{\mathbf{1}} \left( \pm \mathcal{F}_0 \wedge \mathcal{F}_0 + \mathcal{F}_0 \wedge *\mathcal{F}_0 \right) \\ &= \int d^2x \left( (F \pm |\phi|^2)^2 + 2(D_\mu \phi)_a^{(\pm)} (D^\mu \phi)^{(\pm)a} \right), \end{aligned} \tag{3.28}$$

where $F = \frac{1}{2}\epsilon_{\mu\nu}F^{\mu\nu}$, and $(D_\mu\phi)_a^{(\pm)} = \frac{1}{2}[(D_\mu\phi)_a \pm \epsilon_{\mu\nu}\epsilon_{ab}(D^\nu\phi)^b]$. The instanton relations are obtained from the conditions for the absolute minima of the generalized Yang-Mills action:

$$F - |\phi|^2 = 0, \ (D_\mu\phi)_a^{(-)} = 0. \tag{3.29}$$

Now we consider the following "partially" gauge fixed action:

$$\begin{aligned} S = S_0 &+ s\int d^2x \left( \bar{\eta}\partial^\mu \tilde{C}_\mu \right) \\ &+ s\int d^2x \left[ \lambda \left( F - |\phi|^2 - \frac{1}{4}\tilde{\pi} \right) - \chi^{\mu a} \left( (D_\mu\phi)_a^{(-)} + \frac{1}{8}\pi_{\mu a} \right) \right] \\ &+ s\int d^2x \left( -2i\epsilon^{ab}\bar{\eta}\phi_a \tilde{C}_b \right) \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} s\bar{\eta} &= \rho, & s\rho &= 0, \\ s\lambda &= \tilde{\pi}, & s\tilde{\pi} &= 0, \\ s\chi_{\mu a} &= \pi_{\mu a}, & s\pi_{\mu a} &= 0. \end{aligned} \tag{3.31}$$

Here $(\bar{\eta}, \lambda, \chi_{\mu a})$ and $(\rho, \tilde{\pi}, \pi_{\mu a})$ are the anti-ghost fields and auxiliary fields, respectively. And the second, and the third terms in the right-hand-side of (3.30) correspond to the condition of the gauge fixing $\partial^\mu \tilde{C}_\mu = 0$ (the term "partially" means that the degree of freedom for the $SO(2)$ gauge parameter $v$ is unfixed), and the

instanton relations (3.29), respectively. The anti-self-dual fields $\chi_{\mu a}$ and $\pi_{\mu a}$ obey the following conditions:

$$\chi_{\mu a} = -\epsilon_{\mu\nu}\epsilon_{ab}\chi^{\nu b}, \ \ \pi_{\mu a} = -\epsilon_{\mu\nu}\epsilon_{ab}\pi^{\nu b}. \tag{3.32}$$

After integrating $\tilde{\pi}$ and $\pi_{\mu a}$ in (3.30), we obtain the following action:

$$\begin{aligned} S = \int d^2x \Big[ & \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)_a(D^\mu\phi)^a + |\phi|^4 \\ & + i\rho\partial_\mu\tilde{C}^\mu - i\lambda\epsilon^{\mu\nu}\partial_\mu\tilde{C}_\nu \\ & - i\chi_{\mu a}(D^\mu\tilde{C})^a + \partial_\mu\overline{\eta}\partial^\mu\eta \\ & - 2i\epsilon^{ab}\rho\phi_a\tilde{C}_b - 2i\lambda\phi_a\tilde{C}^a - 2i\chi_{\mu a}\epsilon^{ab}\tilde{C}^\mu\phi_b \\ & - \frac{i}{4}\epsilon^{\mu\nu}\chi_{\mu a}\chi_\nu{}^a\eta + 2i\overline{\eta}\epsilon_{ab}\tilde{C}^a\tilde{C}^b + 4\eta\overline{\eta}|\phi|^2 \Big]. \end{aligned} \tag{3.33}$$

This action is invariant under the $SO(2)$ gauge transformation:

$$\begin{aligned} \delta_{gauge}(\phi_a, \tilde{C}_a, \chi_{\mu a}) &= 2v\epsilon_{ab}(\phi^b, \tilde{C}^b, \chi_\mu{}^b), \\ \delta_{gauge}\omega_\mu &= \partial_\mu v, \\ \delta_{gauge}(\tilde{C}_\mu, \rho, \lambda, \eta, \overline{\eta}) &= 0, \end{aligned} \tag{3.34}$$

where $v$ is the gauge parameter.

Furthermore the action (3.33) has the fermionic symmetries $s$, $s_\mu$, and $\tilde{s}$ defined in Table 4. These transformation laws construct the on-shell twisted N=2 super-

| $\phi^A$ | $s\phi^A$ | $s_\mu\phi^A$ | $\widetilde{s}\phi^A$ |
|---|---|---|---|
| $\phi_a$ | $-\tilde{C}_a$ | $\frac{1}{2}\chi_{\mu a}$ | $-\epsilon_{ab}\tilde{C}^b$ |
| $\chi_{\nu a}$ | $4i(D_\nu\phi)_a^{(-)}$ | $-4i(\delta_{\mu\nu}\epsilon_{ab} - \epsilon_{\mu\nu}\delta_{ab})\overline{\eta}\phi^b$ | $-4i\epsilon_{\nu\rho}(D^\rho\phi)_a^{(-)}$ |
| $\omega_\nu$ | $\tilde{C}_\nu$ | $-\frac{1}{2}(\epsilon_{\mu\nu}\lambda + \delta_{\mu\nu}\rho)$ | $\epsilon_{\nu\rho}\tilde{C}^\rho$ |
| $\lambda$ | $-2i(F - \vert\phi\vert^2)$ | $2i\epsilon_{\mu\rho}\partial^\rho\overline{\eta}$ | $0$ |
| $\tilde{C}_a$ | $-2i\epsilon_{ab}\phi^b\eta$ | $-2i(D_\mu\phi)_a^{(+)}$ | $-2i\phi_a\eta$ |
| $\overline{\eta}$ | $\rho$ | $0$ | $-\lambda$ |
| $\tilde{C}_\nu$ | $i\partial_\nu\eta$ | $i(F_{\mu\nu} + \epsilon_{\mu\nu}\vert\phi\vert^2)$ | $-i\epsilon_{\nu\rho}\partial^\rho\eta$ |
| $\rho$ | $0$ | $2i\partial_\mu\overline{\eta}$ | $-2i(F - \vert\phi\vert^2)$ |
| $\eta$ | $0$ | $2\tilde{C}_\mu$ | $0$ |

Table 4: On-shell N=2 twisted super transformation.

symmetry algebra: the operators $s$, $s_\mu$, and $\tilde{s}$ obey the following relations if the

equations of motion obtained from the action (3.33) holds:

$$
\begin{aligned}
s^2 &= i\delta_{gauge\ \eta}, \\
\{s, s_\mu\} &= 2i\partial_\mu - 2i\delta_{gauge\ \omega_\mu}, \\
\{\widetilde{s}, s_\mu\} &= -2i\epsilon_{\mu\nu}\partial^\nu + 2i\delta_{gauge\ \epsilon_{\mu\nu}\omega^\nu}, \\
\widetilde{s}^2 &= i\delta_{gauge\ \eta}, \\
\{s, \widetilde{s}\} &= 0, \\
\{s_\mu, s_\nu\} &= -2i\delta_{\mu\nu}\delta_{gauge\ \overline{\eta}},
\end{aligned}
\tag{3.35}
$$

where $\delta_{gauge\ \phi^A}$ is given in (3.34).

The ghost field sets $(\rho, \tilde{C}_\mu, \lambda)$ and $(\tilde{C}_a, \chi_{\mu a})$ can construct the Dirac-Kähler fermions as follows:

$$
\psi = \tfrac{1}{2}(\rho + \gamma^\mu \tilde{C}_\mu - \gamma^5\lambda), \tag{3.36}
$$

$$
\chi = \tfrac{1}{2}(-\tilde{C}_1 + \gamma^\mu \chi_{\mu 1} - \gamma^5\tilde{C}_2). \tag{3.37}
$$

Indeed the kinetic terms of these fields in (3.33) can be expressed as

$$
\begin{aligned}
&\int d^2x \Big[ i\rho\partial_\mu\tilde{C}^\mu - i\lambda\epsilon^{\mu\nu}\partial_\mu\tilde{C}_\nu - i\chi_{\mu a}(D^\mu\tilde{C})^a \Big] \\
&\qquad = \int d^2x \Big[ i\mathrm{Tr}(\overline{\psi}\gamma^\mu\partial_\mu\psi) + i\mathrm{Tr}(\overline{\chi}\gamma^\mu\partial_\mu\chi) \Big],
\end{aligned}
\tag{3.38}
$$

where $(\overline{\psi}, \overline{\chi}) = (C\psi^T C^{-1}, C\chi^T C^{-1}) = (\psi^T, \chi^T)$.

Thus the action turns out to be the following action with the Dirac-Kähler fermions $\psi$ and $\chi$:

$$
\begin{aligned}
S = \int d^2x \Big[ &\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)_i(D^\mu\phi)^i + |\phi|^4 \\
&+ i\mathrm{Tr}(\overline{\psi}\gamma^\mu\partial_\mu\psi) + i\mathrm{Tr}(\overline{\chi}\gamma^\mu D_\mu\chi) \\
&- 4i\phi_1\mathrm{Tr}(\overline{\psi}\gamma^5\chi) + 4i\phi_1\mathrm{Tr}(\overline{\psi}\gamma^5\chi\gamma^5) \\
&- i\sqrt{2}A\mathrm{Tr}(\overline{\chi}\gamma^5\chi) + i\sqrt{2}B\mathrm{Tr}(\overline{\chi}\chi\gamma^5) \\
&- \tfrac{1}{2}A\partial^2 A + \tfrac{1}{2}B\partial^2 B + 2(A^2 - B^2)|\phi|^2 \Big],
\end{aligned}
\tag{3.39}
$$

where $D_\mu\chi = \partial_\mu\chi + 2\omega_\mu\chi\gamma^5$, and

$$
\overline{\eta} = \tfrac{1}{2\sqrt{2}}(A - B),\ \eta = \sqrt{2}(A + B). \tag{3.40}
$$

This action is equivalent to N=2 super Yang-Mills with the Abelian Higgs system [41] and also topological Bogomol'nyi theory [42] except for the symmetry breaking potential.

# 4 Twisted N=D=2 superspace formalism

In this section we propose the twisted superspace formalism which will reproduce the off-shell N=D=2 BF model and the super Yang-Mills action of the previous section.

## 4.1 Twisted superspace and superfield

We consider the following group element:

$$G(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = e^{i(-x^\mu P_\mu + \theta Q + \theta^\mu Q_\mu + \tilde{\theta}\tilde{Q})}, \tag{4.1}$$

where all $\theta$'s are anticommuting parameters. Twisted D=N=2 superspace is defined in the parameter space of $(x^\mu, \theta, \theta^\mu, \tilde{\theta})$.

By using the relations (2.6) and the Hausdorff's formula $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$, we can show the following relation:

$$G(0, \xi, \xi^\mu, \tilde{\xi}) G(x^\mu, \theta, \theta^\mu, \tilde{\theta}) = G(x^\mu + a^\mu, \theta + \xi, \theta^\mu + \xi^\mu, \tilde{\theta} + \tilde{\xi}), \tag{4.2}$$

where $a^\mu = \frac{i}{2}\xi\theta^\mu + \frac{i}{2}\xi^\mu\theta + \frac{i}{2}\epsilon^\mu{}_\nu\xi^\nu\tilde{\theta} + \frac{i}{2}\epsilon^\mu{}_\nu\tilde{\xi}\theta^\nu$. This multiplication induces a shift transformation in superspace $(x^\mu, \theta, \theta^\mu, \tilde{\theta})$:

$$(x^\mu, \theta, \theta^\mu, \tilde{\theta}) \to (x^\mu + a^\mu, \theta + \xi, \theta^\mu + \xi^\mu, \tilde{\theta} + \tilde{\xi}). \tag{4.3}$$

This transformation can be generated by the differential operators $Q$, $Q_\mu$, and $\tilde{Q}$:

$$\begin{aligned} Q &= \frac{\partial}{\partial\theta} + \frac{i}{2}\theta^\mu\partial_\mu, \\ Q_\mu &= \frac{\partial}{\partial\theta^\mu} + \frac{i}{2}\theta\partial_\mu - \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu, \\ \tilde{Q} &= \frac{\partial}{\partial\tilde{\theta}} - \frac{i}{2}\theta^\mu\epsilon_{\mu\nu}\partial^\nu. \end{aligned} \tag{4.4}$$

Indeed we find

$$\delta_\xi \begin{pmatrix} x^\mu \\ \theta \\ \theta^\mu \\ \tilde{\theta} \end{pmatrix} = (\xi Q + \xi^\mu Q_\mu + \tilde{\xi}\tilde{Q}) \begin{pmatrix} x^\mu \\ \theta \\ \theta^\mu \\ \tilde{\theta} \end{pmatrix} = \begin{pmatrix} a^\mu \\ \xi \\ \xi^\mu \\ \tilde{\xi} \end{pmatrix}. \tag{4.5}$$

These operators satisfy the following relations:

$$\begin{aligned} &\{Q, Q_\mu\} = i\partial_\mu, \ \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\partial^\nu, \\ &Q^2 = \tilde{Q}^2 = \{Q, \tilde{Q}\} = \{Q_\mu, Q_\nu\} = 0. \end{aligned} \tag{4.6}$$

The general scalar superfields in twisted D=N=2 superspace are defined as the functions of $(x^\mu, \theta, \theta^\mu, \tilde{\theta})$, and can be expanded as follows:

$$
\begin{aligned}
F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) =& \phi(x) + \theta^\mu \phi_\mu(x) + \theta^2 \tilde{\phi}(x) \\
&+ \theta\Big(\psi(x) + \theta^\mu \psi_\mu(x) + \theta^2 \tilde{\psi}(x)\Big) \\
&+ \tilde{\theta}\Big(\chi(x) + \theta^\mu \chi_\mu(x) + \theta^2 \tilde{\chi}(x)\Big) \\
&+ \theta\tilde{\theta}\Big(\lambda(x) + \theta^\mu \lambda_\mu(x) + \theta^2 \tilde{\lambda}(x)\Big),
\end{aligned}
\tag{4.7}
$$

where the leading component $\phi(x)$ can be taken to be not only bosonic but also fermionic.

The transformation law of the superfield $F$ is defined as follows:

$$
\begin{aligned}
\delta_\xi F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) =& \delta_\xi \phi(x) + \theta^\mu \delta_\xi \phi_\mu(x) + \theta^2 \delta_\xi \tilde{\phi}(x) \\
&+ \theta\Big(\delta_\xi \psi(x) + \theta^\mu \delta_\xi \psi_\mu(x) + \theta^2 \delta_\xi \tilde{\psi}(x)\Big) \\
&+ \tilde{\theta}\Big(\delta_\xi \chi(x) + \theta^\mu \delta_\xi \chi_\mu(x) + \theta^2 \delta_\xi \tilde{\chi}(x)\Big) \\
&+ \theta\tilde{\theta}\Big(\delta_\xi \lambda(x) + \theta^\mu \delta_\xi \lambda_\mu(x) + \theta^2 \delta_\xi \tilde{\lambda}(x)\Big) \\
=& (\xi Q + \xi^\mu Q_\mu + \tilde{\xi}\tilde{Q}) F(x^\mu, \theta, \theta^\mu, \tilde{\theta}),
\end{aligned}
\tag{4.8}
$$

where $Q$, $Q_\mu$ and $\tilde{Q}$ are the differential operators (4.4). The transformation laws of the component fields $\phi^A(x) = (\phi(x), \phi_\mu(x), \tilde{\phi}(x), \dots)$ are obtained by comparing coefficients of the same superspace parameters in (4.8) (see Table 5). Those transformation laws lead to the following super-charge algebra:

$$
\begin{aligned}
&\{Q, Q_\mu\} = -i\partial_\mu, \ \{\tilde{Q}, Q_\mu\} = i\epsilon_{\mu\nu}\partial^\nu, \\
&Q^2 = \tilde{Q}^2 = \{Q, \tilde{Q}\} = \{Q_\mu, Q_\nu\} = 0,
\end{aligned}
\tag{4.9}
$$

which are the same of (2.6) with $P_\mu = -i\partial_\mu$. It should be noted that the only difference between the super charge algebra and the corresponding differential operator algebra is a sign difference for the derivative.

Given the transformation laws of the component fields, we can expand the superfield $F$ as follows:

$$
\begin{aligned}
F(x^\mu, \theta, \theta^\mu, \tilde{\theta}) &= e^{\delta_\theta}\phi(x) \\
&= \phi(x) + \delta_\theta \phi(x) + \frac{1}{2}{\delta_\theta}^2 \phi(x) + \frac{1}{3!}{\delta_\theta}^3 \phi(x) + \frac{1}{4!}{\delta_\theta}^4 \phi(x),
\end{aligned}
\tag{4.10}
$$

where $\delta_\theta$ is defined as

$$
\delta_\theta = \theta Q + \theta^\mu Q_\mu + \tilde{\theta}\tilde{Q}. \tag{4.11}
$$

As we have seen, the differential operators in (4.4) generate the shift transformation of superspace induced by left multiplication $G(0, \xi, \xi^\mu, \tilde{\xi})G(x^\mu, \theta, \theta^\mu, \tilde{\theta})$. On

| $\phi^A$ | $Q\phi^A$ | $Q_\mu\phi^A$ | $\tilde{Q}\phi^A$ |
|---|---|---|---|
| $\phi$ | $\psi$ | $\phi_\mu$ | $\chi$ |
| $\phi_\rho$ | $-\psi_\rho - \frac{i}{2}\partial_\rho\phi$ | $-\epsilon_{\mu\rho}\tilde{\phi}$ | $-\chi_\rho + \frac{i}{2}\epsilon_{\rho\sigma}\partial^\sigma\phi$ |
| $\tilde{\phi}$ | $\tilde{\psi} + \frac{i}{2}\epsilon^{\rho\sigma}\partial_\rho\phi_\sigma$ | $0$ | $\tilde{\chi} - \frac{i}{2}\partial^\rho\phi_\rho$ |
| $\psi$ | $0$ | $\psi_\mu - \frac{i}{2}\partial_\mu\phi$ | $\lambda$ |
| $\psi_\rho$ | $-\frac{i}{2}\partial_\rho\psi$ | $-\epsilon_{\mu\rho}\tilde{\psi} + \frac{i}{2}\partial_\mu\phi_\rho$ | $-\lambda_\rho + \frac{i}{2}\epsilon_{\rho\sigma}\partial^\sigma\psi$ |
| $\tilde{\psi}$ | $\frac{i}{2}\epsilon^{\rho\sigma}\partial_\rho\psi_\sigma$ | $-\frac{i}{2}\partial_\mu\tilde{\phi}$ | $\tilde{\lambda} - \frac{i}{2}\partial^\rho\psi_\rho$ |
| $\chi$ | $-\lambda$ | $\chi_\mu + \frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\phi$ | $0$ |
| $\chi_\rho$ | $\lambda_\rho - \frac{i}{2}\partial_\rho\chi$ | $-\epsilon_{\mu\rho}\tilde{\chi} - \frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\phi_\rho$ | $+\frac{i}{2}\epsilon_{\rho\sigma}\partial^\sigma\chi$ |
| $\tilde{\chi}$ | $-\tilde{\lambda} + \frac{i}{2}\epsilon^{\rho\sigma}\partial_\rho\chi_\sigma$ | $\frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\tilde{\phi}$ | $-\frac{i}{2}\partial^\rho\chi_\rho$ |
| $\lambda$ | $0$ | $\lambda_\mu + \frac{i}{2}\partial_\mu\chi + \frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\psi$ | $0$ |
| $\lambda_\rho$ | $-\frac{i}{2}\partial_\rho\lambda$ | $-\epsilon_{\mu\rho}\tilde{\lambda} - \frac{i}{2}\partial_\mu\chi_\rho - \frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\psi_\rho$ | $\frac{i}{2}\epsilon_{\rho\sigma}\partial^\sigma\lambda$ |
| $\tilde{\lambda}$ | $\frac{i}{2}\epsilon^{\rho\sigma}\partial_\rho\lambda_\sigma$ | $\frac{i}{2}\partial_\mu\tilde{\chi} + \frac{i}{2}\epsilon_{\mu\nu}\partial^\nu\tilde{\psi}$ | $-\frac{i}{2}\partial^\rho\lambda_\rho$ |

Table 5: Transformation laws of the component fields of $F$

the other hand, there exist the differential operators which generate the shift transformation induced by right multiplication $G(x^\mu, \theta, \theta^\mu, \tilde{\theta})G(0, \xi, \xi^\mu, \tilde{\xi})$:

$$\begin{aligned} D &= \frac{\partial}{\partial\theta} - \frac{i}{2}\theta^\mu\partial_\mu, \\ D_\mu &= \frac{\partial}{\partial\theta^\mu} - \frac{i}{2}\theta\partial_\mu + \frac{i}{2}\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu, \\ \tilde{D} &= \frac{\partial}{\partial\tilde{\theta}} + \frac{i}{2}\theta^\mu\epsilon_{\mu\nu}\partial^\nu, \end{aligned} \tag{4.12}$$

which satisfy the relations:

$$\begin{aligned} \{D, D_\mu\} = -i\partial_\mu,\ \{\tilde{D}, D_\mu\} = i\epsilon_{\mu\nu}\partial^\nu, \\ D^2 = \tilde{D}^2 = \{D, \tilde{D}\} = \{D_\mu, D_\nu\} = 0, \end{aligned} \tag{4.13}$$

where only the sign of $\partial_\mu$ is changed from the left operator algebra (4.6). $Q^A = (Q, Q_\mu, \tilde{Q})$ and $D^A = (D, D_\mu, \tilde{D})$ anticommute:

$$\{Q^A, D^B\} = 0. \tag{4.14}$$

## 4.2 Chiral decomposition of twisted supercharge

We have defined the general scalar superfields in the previous subsection. However their representations are reducible. We can obtain the irreducible representations by imposing chiral conditions. In this subsection we introduce the twisted chiral differential operators by which superfields can be classified.

Let us first introduce the following chiral generators:

$$Q^\pm = \frac{1}{\sqrt{2}}(Q \mp i\tilde{Q}), \quad Q^\pm{}_\mu = \frac{1}{\sqrt{2}}(Q_\mu \pm i\epsilon_{\mu\nu}Q^\nu). \tag{4.15}$$

As we can see from (2.6), these new generators satisfy the following relations:

$$\begin{aligned} Q^{\pm 2} &= 0, & \{Q^+, Q^-\} &= 0, \\ \{Q^\pm, Q^\pm{}_\mu\} &= \delta^\pm_{\mu\nu} P^\nu, & \{Q^\pm, Q^\mp{}_\mu\} &= 0, \\ \{Q^\pm{}_\mu, Q^\pm{}_\nu\} &= 0, & \{Q^+{}_\mu, Q^-{}_\nu\} &= 0, \end{aligned} \tag{4.16}$$

where

$$\delta^\pm_{\mu\nu} = \delta_{\mu\nu} \pm i\epsilon_{\mu\nu}. \tag{4.17}$$

The super-transformation $\delta_\theta$ can be represented by the generators $Q^+$, $Q^+{}_\mu$, $Q^-$, and $Q^-{}_\mu$:

$$\delta_\theta = \theta Q + \theta^\mu Q_\mu + \tilde{\theta}\tilde{Q} = \theta^- Q^+ + \tfrac{1}{2}\theta^{-\mu} Q^+{}_\mu + \theta^+ Q^- + \tfrac{1}{2}\theta^{+\mu} Q^-{}_\mu, \tag{4.18}$$

where

$$\theta^\pm = \frac{1}{\sqrt{2}}(\theta \mp i\tilde{\theta}), \quad \theta^\pm{}_\mu = \frac{1}{\sqrt{2}}(\theta_\mu \pm i\epsilon_{\mu\nu}\theta^\nu). \tag{4.19}$$

These new parameters satisfy the following properties:

$$(\theta^\pm)^* = \theta^\mp, \qquad (\theta^\pm{}_\mu)^* = \theta^\mp{}_\mu, \tag{4.20}$$

where $\theta^{+\mu}$ and $\theta^{-\mu}$ are self-dual and anti-self-dual parameters, respectively, in the following sense:

$$\theta^{\pm\mu} = \pm i\epsilon^{\mu\nu}\theta^\pm{}_\nu. \tag{4.21}$$

Now we redefine superspace by newly defined fermionic chiral parameters $(x^\mu, \theta^+, \theta^{+\mu}, \theta^-, \theta^{-\mu})$. The differential operators corresponding to the chiral generators defined in (4.15) are obtained from (4.4)

$$\begin{aligned} Q^\mp &= \frac{\partial}{\partial\theta^\pm} + \frac{i}{4}\theta^{\pm\mu}\partial^\mp{}_\mu, \\ Q^\mp{}_\mu &= \frac{\partial}{\partial\theta^{\pm\mu}} + \frac{i}{2}\theta^\pm\partial^\mp{}_\mu, \end{aligned} \tag{4.22}$$

which satisfy the following operator relations:

$$\begin{aligned} Q^{\pm 2} &= 0, & \{Q^+, Q^-\} &= 0, \\ \{Q^\pm, Q^\pm{}_\mu\} &= i\partial^\pm{}_\mu, & \{Q^\pm, Q^\mp{}_\mu\} &= 0, \\ \{Q^\pm{}_\mu, Q^\pm{}_\nu\} &= 0, & \{Q^+{}_\mu, Q^-{}_\nu\} &= 0, \end{aligned} \tag{4.23}$$

with

$$\partial^\pm{}_\mu = \partial_\mu \pm i\epsilon_{\mu\nu}\partial^\nu. \tag{4.24}$$

It is worth to note the following relations with the current conventions:

$$\begin{aligned} (Q^\mp)^* &= -Q^\pm, \quad (Q^\mp_\mu)^* = -Q^\pm_\mu \\ \left(\frac{\partial}{\partial\theta^\pm}\right)^* &= -\frac{\partial}{\partial\theta^\mp}, \quad \left(\frac{\partial}{\partial\theta^{\pm\mu}}\right)^* = -\frac{\partial}{\partial\theta^{\mp\mu}} \\ \partial^\pm_\mu x^\nu &= \frac{\partial}{\partial\theta^{\pm\mu}}\theta^{\pm\nu} = \delta^\pm{}_\mu{}^\nu, \end{aligned} \tag{4.25}$$

where $\delta^{\pm}_{\mu\nu}$ is defined in (4.17).

The differential operators which generate the shift from right multiplication are obtained from (4.12)

$$\begin{aligned} D^{\mp} &\equiv \frac{1}{\sqrt{2}}(D \pm i\tilde{D}) = \frac{\partial}{\partial\theta^{\pm}} - \frac{i}{4}\theta^{\pm\mu}\partial^{\mp}{}_{\mu}, \\ D^{\mp}{}_{\mu} &\equiv \frac{1}{\sqrt{2}}(D_{\mu} \mp i\epsilon_{\mu\nu}D^{\nu}) = \frac{\partial}{\partial\theta^{\pm\mu}} - \frac{i}{2}\theta^{\pm}\partial^{\mp}{}_{\mu}, \end{aligned} \tag{4.26}$$

which satisfy

$$\begin{aligned} D^{\pm 2} &= 0, & \{D^{+}, D^{-}\} &= 0, \\ \{D^{\pm}, D^{\pm}{}_{\mu}\} &= -i\partial^{\pm}{}_{\mu}, & \{D^{\pm}, D^{\mp}{}_{\mu}\} &= 0, \\ \{D^{\pm}{}_{\mu}, D^{\pm}{}_{\nu}\} &= 0, & \{D^{+}{}_{\mu}, D^{-}{}_{\nu}\} &= 0. \end{aligned} \tag{4.27}$$

The differential operators $Q'^{A} = (Q^{\pm}, Q^{\pm\mu})$ and $D'^{A} = (D^{\pm}, D^{\pm\mu})$ anticommute:

$$\{Q'^{A}, D'^{B}\} = 0. \tag{4.28}$$

## 4.3 Chiral superfields

We can classify the chiral superfields into four types which are constrained by the following four types of conditions:

$$D^{+}\Psi = D^{-}\Psi = 0, \tag{4.29}$$
$$D^{+}{}_{\mu}\overline{\Psi} = D^{-}{}_{\mu}\overline{\Psi} = 0, \tag{4.30}$$
$$D^{+}{}_{\mu}\Phi = D^{-}\Phi = 0, \tag{4.31}$$
$$D^{+}\Phi^{+} = D^{-}{}_{\mu}\Phi^{+} = 0. \tag{4.32}$$

Here in this subsection we consider that the superfield has single component and thus all the component fields have abelian nature.

### 4.3.1 $\Psi$ and $\overline{\Psi}$

Firstly, we consider the chiral superfields characterized by the conditions (4.29) and (4.30).

From the definition (4.26) we see that the condition (4.29) is equivalent to the condition

$$D\Psi = \tilde{D}\Psi = 0. \tag{4.33}$$

The superfields satisfying the above condition are functions of

$$z^{\mu} = x^{\mu} + \frac{i}{2}\theta\theta^{\mu} - \frac{i}{2}\epsilon^{\mu}{}_{\nu}\theta^{\nu}\tilde{\theta} \tag{4.34}$$

and $\theta^\mu$ since

$$D z^\mu = D\theta^\mu = 0, \quad \tilde{D} z^\mu = \tilde{D}\theta^\mu = 0. \tag{4.35}$$

In general we have the following relation:

$$\begin{aligned} e^{\theta^\mu Q_\mu} e^{\theta Q + \tilde{\theta}\tilde{Q}} \varphi(x^\mu) &= e^{\delta_\theta - \frac{1}{2}\theta\theta^\mu i\partial_\mu + \frac{1}{2}\tilde{\theta}\theta^\mu i\epsilon_{\mu\nu}\partial^\nu} \varphi(x^\mu) \\ &= e^{\delta_\theta}\varphi\left(x^\mu - \frac{i}{2}\theta\theta^\mu + \frac{i}{2}\epsilon^{\mu\nu}\theta_\nu\tilde{\theta}\right), \end{aligned} \tag{4.36}$$

and thus

$$e^{\delta_\theta}\varphi(x^\mu) = e^{\theta^\mu Q_\mu} e^{\theta Q + \tilde{\theta}\tilde{Q}} \varphi(z^\mu), \tag{4.37}$$

where $\delta_\theta$ is given by (4.11). Therefore the chiral superfield $\Psi$ can be expanded as follows:

$$\begin{aligned} \Psi &= \Psi(z^\mu, \theta^\mu) = e^{\delta_\theta}\phi(x) = e^{\theta^\mu Q_\mu}\phi(z) \\ &= \phi(z) + \theta^\mu\phi_\mu(z) + \theta^2\tilde{\phi}(z) \\ &= \phi(x) + \theta^\mu\phi_\mu(x) \\ &\quad + \tfrac{i}{2}\theta\theta^\mu\partial_\mu\phi(x) + \theta^2\tilde{\phi}(x) - \tfrac{i}{2}\epsilon^\mu{}_\nu\theta^\nu\tilde{\theta}\partial_\mu\phi(x) \\ &\quad + \tfrac{i}{2}\theta\theta^2\epsilon^{\mu\nu}\partial_\mu\phi_\nu - \tfrac{i}{2}\theta^2\tilde{\theta}\partial^\mu\phi_\mu + \tfrac{1}{4}\theta^4\partial^2\phi(x), \end{aligned} \tag{4.38}$$

where $\theta^2 = \frac{1}{2}\epsilon_{\mu\nu}\theta^\mu\theta^\nu$ and $\theta^4 = \theta\tilde{\theta}\theta^2$. The set $(\phi, \phi_\mu, \tilde{\phi})$ constructs twisted N=2 off-shell super multiplet(see Table 6).

| $\phi^A$ | $Q\phi^A$ | $Q_\mu\phi^A$ | $\tilde{Q}\phi^A$ |
|---|---|---|---|
| $\phi$ | $0$ | $\phi_\mu$ | $0$ |
| $\phi_\rho$ | $-i\partial_\rho\phi$ | $-\epsilon_{\mu\rho}\tilde{\phi}$ | $i\epsilon_{\rho\nu}\partial^\nu\phi$ |
| $\tilde{\phi}$ | $i\epsilon^{\rho\sigma}\partial_\rho\phi_\sigma$ | $0$ | $-i\partial^\rho\phi_\rho$ |
| $\psi$ | $\chi$ | $0$ | $\tilde{\chi}$ |
| $\chi$ | $0$ | $-i\partial_\mu\psi$ | $\tilde{\psi}$ |
| $\tilde{\chi}$ | $-\tilde{\psi}$ | $i\epsilon_{\mu\nu}\partial^\nu\psi$ | $0$ |
| $\tilde{\psi}$ | $0$ | $i\epsilon_{\mu\nu}\partial^\nu\chi + i\partial_\mu\tilde{\chi}$ | $0$ |

Table 6: Chiral twisted super multiplet $(\phi, \phi_\mu, \tilde{\phi})$ and $(\psi, \chi, \tilde{\chi}, \tilde{\psi})$.

On the other hand we can see from (4.26), the conditions in (4.30) are equivalent to the condition

$$D_\mu\overline{\Psi} = 0. \tag{4.39}$$

This type of chiral superfields are the functions of

$$\tilde{z}^\mu = x^\mu - \frac{i}{2}\theta\theta^\mu + \frac{i}{2}\epsilon^\mu{}_\nu\theta^\nu\tilde{\theta}, \tag{4.40}$$

and $\theta$, $\tilde{\theta}$ since

$$D_\mu\tilde{z}^\nu = 0, \quad D_\mu\theta = D_\mu\tilde{\theta} = 0. \tag{4.41}$$

Then $\overline{\Psi}$ can be expanded as follows:

$$\begin{aligned}\overline{\Psi} &= \overline{\Psi}(\tilde{z}^\mu, \theta, \tilde{\theta}) = e^{\delta_\theta}\psi(x) = e^{\theta Q + \tilde{\theta}\tilde{Q}}\psi(\tilde{z}) \\ &= \psi(\tilde{z}) + \theta\chi(\tilde{z}) + \tilde{\theta}\tilde{\chi}(\tilde{z}) + \theta\tilde{\theta}\tilde{\psi}(\tilde{z}) \\ &= \psi(x) + \theta\chi + \tilde{\theta}\tilde{\chi} \\ &\quad + \theta\tilde{\theta}\tilde{\psi}(x) - \tfrac{i}{2}\theta\theta^\mu\partial_\mu\psi(x) + \tfrac{i}{2}\epsilon^\mu{}_\nu\theta^\nu\tilde{\theta}\partial_\mu\psi(x) \\ &\quad - \tfrac{i}{2}\theta\theta^\mu\tilde{\theta}(\epsilon_{\mu\nu}\partial^\nu\chi + \partial_\mu\tilde{\chi}) + \tfrac{1}{4}\theta^4\partial^2\psi(x).\end{aligned} \tag{4.42}$$

The set $(\psi, \chi, \tilde{\chi}, \tilde{\psi})$ also constructs a super multiplet(see Table 6).

We now introduce off-shell N=2 supersymmetric action:

$$S = \int d^2x \int d^4\theta \Big( i^{\epsilon_\Psi}\overline{\Psi}(x^\mu, \theta, \theta^\mu, \tilde{\theta})\Psi(x^\mu, \theta, \theta^\mu, \tilde{\theta})\Big). \tag{4.43}$$

We can take the chiral superfields to be not only bosonic but also fermionic. $\epsilon_\Psi$ should be taken 0 or 1 for bosonic or fermionic $(\Psi, \overline{\Psi})$, respectively.

For fermionic $(\Psi, \overline{\Psi})$, the fields in the expansion of the superfield (4.38) and (4.42) can be renamed as:

$$\begin{aligned}\Psi &= \Psi(z^\mu, \theta^\mu) = ie^{\theta^\mu Q_\mu}c(z) = ic(z) + \theta^\mu\omega_\mu(z) + i\theta^2\lambda(z) \\ \overline{\Psi} &= \overline{\Psi}(\tilde{z}^\mu, \theta, \tilde{\theta}) = ie^{\theta Q + \tilde{\theta}\tilde{Q}}\overline{c}(\tilde{z}) = i\overline{c}(\tilde{z}) + \theta b(\tilde{z}) + \tilde{\theta}\phi(\tilde{z}) - i\theta\tilde{\theta}\rho(\tilde{z}),\end{aligned} \tag{4.44}$$

where we have the correspondence of the fields; $(\psi, \chi, \tilde{\chi}, \tilde{\psi}) \to (i\overline{c}, b, \phi, -i\rho)$ and $(\phi, \phi_\mu, \tilde{\phi}) \to (ic, \omega_\mu, i\lambda)$. Then the action (4.43) leads

$$\begin{aligned}S_f &= \int d^2x \int d^4\theta\ (i\overline{\Psi}\Psi) = \int d^2x \int d^4\theta\ e^{\delta_\theta}(-i\bar{c}c) \\ &= \int_{M_2} d^2x\ Q\tilde{Q}\frac{1}{2}\epsilon^{\mu\nu}Q_\mu Q_\nu(-i\bar{c}c) \\ &= \int d^2x \Big(\phi\epsilon^{\mu\nu}\partial_\mu\omega_\nu + b\partial^\mu\omega_\mu + i\partial_\mu\bar{c}\partial^\mu c - i\lambda\rho\Big),\end{aligned} \tag{4.45}$$

where $\delta_\theta$ is defined in (4.11). This action is exactly the same as the quantized BF action (2.14) which is off-shell N=2 supersymmetric. The off-shell N=2 supersymmetry transformations of the fields in (4.44) is given in Table 6 and is exactly the same as those of Table 2.

As we have already pointed out that the simple form of the action (2.15) has suggested a hidden mechanism of a superspace formalism. We have in fact found the twisted N=2 superspace formalism which reproduces the quantized BF action up to the last term $-i\lambda\rho$ which includes auxiliary fields and is crucial to fufill off-shell N=2 supersymmetry. We find the following correspondence:

$$s = Q, \quad s_\mu = Q_\mu, \quad \tilde{s} = \tilde{Q}. \tag{4.46}$$

where $s = Q$ can be regarded as BRST charge associated with the gauge symmetry $\delta_{gauge}\omega_\mu = \partial_\mu v$ with the Landau type gauge fixing, $\partial_\mu \omega^\mu = 0$. Integrating $\lambda$ and $\rho$ out, the action (4.45) coincides exactly with the quantized BF action (2.10), where the off-shell N=2 supersymmetry reduces to on-shell N=2 supersymmetry.

For bosonic $(\Psi, \overline{\Psi})$ the action (4.43) can be written as follows:

$$
\begin{aligned}
S_b &= \int d^2x \int d^4\theta\ (\overline{\Psi}\Psi) = \int d^2x \int d^4\theta\ e^{\delta_\theta}(\psi\phi) \\
&= \int_{M_2} d^2x\ Q\tilde{Q}\frac{1}{2}\epsilon^{\mu\nu}Q_\mu Q_\nu(\psi\phi) \\
&= \int d^2x\Big(i\tilde{\chi}\epsilon^{\mu\nu}\partial_\mu\phi_\nu + i\chi\partial^\mu\phi_\mu - \partial^\mu\psi\partial_\mu\phi + \tilde{\psi}\tilde{\phi}\Big),
\end{aligned}
\tag{4.47}
$$

where $(\phi, \psi, \tilde{\phi}, \tilde{\psi})$ and $(\phi_\mu, \chi, \tilde{\chi})$ are bosonic and fermionic fields, respectively. The fermionic terms in (4.47) change into matter fermions via Dirac-Kähler fermion mechanism:

$$
\int d^2x\ \ (i\tilde{\chi}\epsilon^{\mu\nu}\partial_\mu\phi_\nu + i\chi\partial^\mu\phi_\mu) = \int d^2x\ \mathrm{Tr}\left(i\overline{\xi}\gamma^\mu\partial_\mu\xi\right) \tag{4.48}
$$

where the Dirac-Kähler fermion $\xi$ is defined as

$$
\xi_{\alpha\beta} = \frac{1}{2}\left(\mathbf{1}\chi + \gamma^\mu\phi_\mu + \gamma^5\tilde{\chi}\right)_{\alpha\beta}, \tag{4.49}
$$

with $\overline{\xi} = C\xi^T C = \xi^T$. We can recognize that each spinor suffix of this Dirac-Kähler fermion has the Majorana Weyl fermion nature. We now redefine the bosonic fields as follows:

$$
\begin{aligned}
\phi_0 &= \frac{1}{2}(\phi + \psi), \phi_1 = \frac{1}{2}(\phi - \psi), \\
F_0 &= \frac{1}{2}(\tilde{\phi} + \tilde{\psi}), F_1 = \frac{1}{2}(\tilde{\phi} - \tilde{\psi}).
\end{aligned}
\tag{4.50}
$$

Then the action (4.47) can be rewritten by the new fields:

$$
S_b = \int d^2x \sum_{i=1}^{2}\Big(i\xi_\alpha{}^i\gamma^\mu{}_{\alpha\beta}\partial_\mu\xi_\beta{}^i + \partial_\mu\phi^i\partial^\mu\phi^i - F^iF^i\Big), \tag{4.51}
$$

where we further redefine $i\phi_0 \to \phi_2$ and $iF_0 \to F_2$ [3]. This is the 2-dimensional version of N=2 Wess-Zumino action which has off-shell N=2 supersymmetry invariance. It is important to recognize at this stage that the second suffix of the Dirac-Kähler matter fermion in the action is N=2 extended supersymmetry suffix.

[3] In general the ghosts carry the indefinite metric and thus have non-unitary nature while this type of supersymmetric model is unitary. This change of sings for the quadratic terms can be related with this fact[51].

This shows that the fermionic and bosonic chiral bi-linear form of the superfield actions lead an extended version of the quantized BF action and Wess-Zumino action, respectively, which have off-shell N=2 supersymmetry invariance. It is interesting to note that this extended quantized BF action (4.45) can be transformed to the D=N=2 Wess-Zumino action. To see this, we make a chain of redefinitions of fields, $\{\phi, \tilde{\phi}\} \to \{\phi, \tilde{\rho}\} \to \{\psi_\mu\}$ and $\{\chi, \phi_\mu, \tilde{\chi}\} \to \{\chi, \alpha, \beta, \tilde{\chi}\} \to \{\chi, f, \beta, \tilde{\chi}\} \to \{\phi_0, \phi_1, F_0, F_1\}$:

$$\begin{aligned} &\tilde{\phi} = -\partial^2 \tilde{\rho}, \;\; \psi_\mu = \partial_\mu \phi + \epsilon_{\mu\nu}\partial^\nu \tilde{\rho}, \\ &\phi_\mu = \partial_\mu \alpha + \epsilon_{\mu\nu}\partial^\nu \beta, \;\; f = \partial^2 \alpha, \; \left\{ \begin{array}{l} \tilde{\chi} = \phi_0 + \phi_1 \\ \beta = \phi_0 - \phi_1 \end{array} \right. , \; \left\{ \begin{array}{l} \chi = F_0 + F_1 \\ f = F_0 - F_1. \end{array} \right. \end{aligned} \tag{4.52}$$

Then the action (4.45) can be rewritten as the following Wess-Zumino action:

$$S_f = \int d^2x \sum_{i=1}^{2} \Big( i{\psi_\alpha}^i {\gamma^\mu}_{\alpha\beta} \partial_\mu {\psi_\beta}^i + \partial_\mu \phi^i \partial^\mu \phi^i - F^i F^i \Big), \tag{4.53}$$

where we again redefine $i\phi_0 \to \phi_2$ and $iF_0 \to F_1$, and the Dirac-Kähler matter fermion $\psi_{\alpha\beta}$ is defined as

$$\psi_{\alpha\beta} = \frac{1}{2}\left( \mathbf{1}\psi + \gamma^\mu \psi_\mu + \gamma^5 \tilde{\psi} \right)_{\alpha\beta}, \tag{4.54}$$

where we have identified the second suffix of the Dirac-Kähler fermion as N=2 supersymmetry suffix.

### 4.3.2 Non-abelian Extension

In this subsection we extend the abelian version of twisted superspace formulation of the previous subsection into non-abelian case. We point out that the extension from the abelian to a non-abelian version is straightforward, however, the chiral structure is sacrificed by the non-abelian nature. Using the general relation (4.37), we obtain the following superfield generated from the non-abelian ghost field:

$$\Psi = e^{\delta_\theta} c(x^\mu) = e^{\theta^\mu s_\mu} e^{\theta s + \tilde{\theta}\tilde{s}} c(z^\mu) \tag{4.55}$$

$$\quad = e^{\theta^\mu s_\mu}(c + \theta(-c^2)), \tag{4.56}$$

where we have introduced BRST transformation of the non-abelian ghost field: $sc = -c^2$ in Table 3. Thus the superfield $\Psi$ is the function of $z^\mu, \theta_\mu$ and $\theta$, and thus $\Psi$ is not twisted chiral superfield anymore since it includes $\theta$ and thus

$$D\Psi \neq 0, \quad \tilde{D}\Psi = 0. \tag{4.57}$$

We can yet expand the superfield $\Psi$ in the following form:

$$\begin{aligned}
\Psi &= c(z) - i\theta^\mu\omega_\mu(z) + \theta^2\lambda(z) - \theta c^2(z) + i\theta\theta^\mu[\omega_\mu(z), c(z)] \\
&\quad -\theta\theta^2\left(\{\lambda(z), c(z)\} - \frac{1}{2}\epsilon^{\mu\nu}[\omega_\mu(z), \omega_\nu(z)]\right) \\
&= c(x) + \frac{i}{2}\theta\theta^\mu\partial_\mu c(x) + \frac{i}{2}\theta^\mu\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu c(x) + \frac{1}{4}\theta^4\partial^2 c(x) - \theta c^2(x) - \frac{i}{2}\theta\theta^\mu\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu c^2(x) \\
&\quad -i\theta^\mu\omega_\mu(x) - \frac{1}{2}\theta^2\tilde{\theta}\partial^\mu\omega_\mu(x) + \frac{1}{2}\theta\theta^2\epsilon^{\mu\nu}\partial_\mu\omega_\nu(x) + \theta^2\lambda(x) + i\theta\theta^\mu[\omega_\mu(x), c(x)] \\
&\quad +\frac{1}{2}\theta^4\partial^\mu[\omega_\mu(x), c(x)] - \theta\theta^2(\{\lambda(x), c(x)\} - \epsilon^{\mu\nu}[\omega_\mu(x), \omega_\nu(x)]).
\end{aligned} \tag{4.58}$$

Similarly we can construct the superfield $\overline{\Psi}$ generated from the nob-abelian anti-ghost field:

$$\overline{\Psi} = e^{\delta_\theta}\overline{c}(x^\mu) = e^{\theta s + \tilde{\theta}\tilde{s}}\ e^{\theta^\mu s_\mu}\ \overline{c}(\tilde{z}^\mu) = e^{\theta s + \tilde{\theta}\tilde{s}}\ \overline{c}(\tilde{z}^\mu), \tag{4.59}$$

which satisfies twisted anti-chiral condition (4.39). We can expand the superfield $\overline{\Psi}$ in the following form:

$$\begin{aligned}
\overline{\Psi} &= \overline{c}(z) - i\theta b(z) - i\tilde{\theta}\phi(z) + i\theta\tilde{\theta}\ ([\phi(z), c(z)] + i\rho(z)) \\
&= \overline{c}(x) - \frac{i}{2}\theta\theta^\mu\partial_\mu\overline{c}(x) - \frac{i}{2}\theta^\mu\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu\overline{c}(x) + \frac{1}{4}\theta^4\partial^2\overline{c}(x) - i\theta b(x) - \frac{1}{2}\theta\theta^\mu\tilde{\theta}\epsilon_{\mu\nu}\partial^\nu b(x) \\
&\quad -i\tilde{\theta}\phi(x) - \frac{1}{2}\theta\theta^\mu\tilde{\theta}\partial_\mu\phi(x) + i\theta\tilde{\theta}\ ([\phi(x), c(x)] + i\rho(x)).
\end{aligned} \tag{4.60}$$

Even though $\Psi$ is not twisted chiral superfield we can still construct the off-shell N=2 twisted super symmetric non-abelian BF action just like the abelian case:

$$\begin{aligned}
S_{\text{off-shell NABQBF}} &= \int d^2x \int d^4\theta\ \mathrm{Tr}(i\overline{\Psi}\Psi) = \int d^2x \int d^4\theta\ e^{\delta_\theta}\mathrm{Tr}(-i\overline{c}c) \\
&= \int d^2x\ s\tilde{s}\frac{1}{2}\epsilon^{\mu\nu}s_\mu s_\nu \mathrm{Tr}(-i\overline{c}c) \\
&= \int d^2x\ \mathrm{Tr}[\phi F + b\partial^\mu\omega_\mu + i\partial^\mu\overline{c}D_\mu c - i\lambda\rho].
\end{aligned} \tag{4.61}$$

### 4.3.3 $\Phi$ and $\Phi^+$

We consider chiral superfields constrained by the conditions (4.31)

$$D^-\Phi = D^+{}_\mu\Phi = 0, \tag{4.62}$$

where the gauge algebra is not introduced here. This type of chiral superfields are the functions of $y^\mu = x^\mu + \frac{i}{2}\theta^+\theta^{+\mu} - \frac{i}{2}\theta^-\theta^{-\mu}$, $\theta^{+\mu}$, and $\theta^-$, since

$$\begin{aligned}
D^-y^\mu &= D^-\theta^{+\mu} = D^-\theta^- = 0, \\
D^+_\mu y^\nu &= D^+_\mu\theta^{+\nu} = D^+_\mu\theta^- = 0.
\end{aligned} \tag{4.63}$$

Then the superfield can be expanded as follows:

$$
\begin{aligned}
\Phi &= \Phi(y^\mu, \theta^{+\mu}, \theta^-) \\
&= e^{\theta^- Q^+ + \frac{1}{2}\theta^{+\mu} Q^-_\mu} \phi(y) \\
&= \phi(y) + \frac{1}{2}\theta^{+\mu}\xi^-{}_\mu(y) + \theta^- \left( \xi(y) + \frac{1}{2}\theta^{+\mu} K^-{}_\mu(y) \right) \\
&= \phi(x) + \frac{1}{2}\theta^{+\mu}\xi^-{}_\mu(x) + \theta^-\xi(x) \\
&\quad + \frac{i}{4}\theta^+\theta^{+\mu}\partial^-{}_\mu\phi(x) - \frac{i}{4}\theta^-\theta^{-\mu}\partial^+{}_\mu\phi(x) + \frac{1}{2}\theta^-\theta^{+\mu}K^-{}_\mu(x) \\
&\quad + \frac{i}{8}\theta^-\theta^{+\rho}\theta^-{}_\rho\partial^{+\mu}\xi^-_\mu(x) + \frac{i}{4}\theta^+\theta^{+\mu}\theta^-\partial^-{}_\mu\xi(x) \\
&\quad - \frac{1}{8}\theta^+\theta^-\theta^{+\mu}\theta^-{}_\mu\partial^2\phi(x),
\end{aligned}
\tag{4.64}
$$

where $\xi^-{}_\mu$ and $K^-{}_\mu$ are anti-self-dual. The N=2 supersymmetry transformation for the component fields $\phi$, $\xi^-{}_\mu$, $\xi$, and $K^-{}_\mu$ can be obtained by operating the differential operators (4.22) to this representation (see Table 7).

Another type of chiral super fields $\Phi^+$ can be obtained by taking a conjugation of $\Phi$,

$$
\begin{aligned}
\Phi^+ = \Phi^* &= \phi^*(x) - \frac{1}{2}\theta^{-\mu}\xi^+{}_\mu(x) - \theta^+\xi^*(x) \\
&\quad - \frac{i}{4}\theta^+\theta^{+\mu}\partial^-{}_\mu\phi^*(x) + \frac{i}{4}\theta^-\theta^{-\mu}\partial^+{}_\mu\phi^*(x) - \frac{1}{2}\theta^+\theta^{-\mu}K^+{}_\mu(x) \\
&\quad + \frac{i}{8}\theta^+\theta^{+\rho}\theta^-{}_\rho\partial^{-\mu}\xi^+{}_\mu(x) + \frac{i}{4}\theta^+\theta^{-\mu}\theta^-\partial^+{}_\mu\xi^*(x) \\
&\quad - \frac{1}{8}\theta^+\theta^-\theta^{+\mu}\theta^-{}_\mu\partial^2\phi^*(x) \\
&= \phi^*(y^+) - \frac{1}{2}\theta^{-\mu}\xi^+{}_\mu(y^+) + \theta^+\left( -\xi^*(y^+) - \frac{1}{2}\theta^{-\mu}K^+{}_\mu(y^+) \right) \\
&= \Phi^+(y^{+\mu}, \theta^{-\mu}, \theta^+) = e^{\theta^+ Q^- + \frac{1}{2}\theta^{-\mu}Q^+_\mu}\phi(y^+),
\end{aligned}
\tag{4.65}
$$

where $\xi^+{}_\mu = (\xi^-{}_\mu)^*$, $K^+{}_\mu = (K^-{}_\mu)^*$, and $y^{+\mu} = x^\mu - \frac{i}{2}\theta^+\theta^{+\mu} + \frac{i}{2}\theta^-\theta^{-\mu}$. Indeed $\Phi^+$ satisfy the following conditions:

$$
D^+\Phi^+ \quad = \quad D^-{}_\mu\Phi^+ = 0, \tag{4.66}
$$

since

$$
\begin{aligned}
D^+ y^{+\mu} &= D^+\theta^+ = D^+\theta^{-\mu} = 0 \\
D^-_\mu y^{+\nu} &= D^-_\mu\theta^+ = D^-_\mu\theta^{-\nu} = 0.
\end{aligned}
\tag{4.67}
$$

Similarly we can find N=2 supersymmetry transformation laws of the component fields $\phi^*$, $\xi^+{}_\mu$, $\xi^*$, and $K^+{}_\mu$(see Table 7).

| $\phi^A$ | $Q^-\phi^A$ | $Q^-{}_\mu\phi^A$ | $Q^+\phi^A$ | $Q^+{}_\mu\phi^A$ |
|---|---|---|---|---|
| $\phi$ | $0$ | $\xi^-{}_\mu$ | $\xi$ | $0$ |
| $\xi^-{}_\rho$ | $-i\partial^-{}_\rho\phi$ | $0$ | $-K^-{}_\rho$ | $0$ |
| $\xi$ | $0$ | $K^-{}_\mu$ | $0$ | $-i\partial^+{}_\mu\phi$ |
| $K^-{}_\rho$ | $-i\partial^-{}_\rho\xi$ | $0$ | $0$ | $i\partial^+{}_\mu\xi^-{}_\rho$ |
| $\phi^*$ | $-\xi^*$ | $0$ | $0$ | $-\xi^+{}_\mu$ |
| $\xi^+{}_\rho$ | $-K^+{}_\rho$ | $0$ | $i\partial^+{}_\rho\phi^*$ | $0$ |
| $\xi^*$ | $0$ | $i\partial^-{}_\mu\phi^*$ | $0$ | $K^+{}_\mu$ |
| $K^+{}_\rho$ | $0$ | $i\partial^-{}_\mu\xi^+{}_\rho$ | $-i\partial^+{}_\rho\xi^*$ | $0$ |

Table 7: N=2 super-transformation of chiral twisted super multiplet. $(\phi, \xi^-{}_\rho, \xi, K^-{}_\rho)$ and ($\phi^*$, $\xi^+{}_\rho$, $\xi^*$, and $K^+{}_\rho$).

Since $\theta^4$ component of $\Phi^+\Phi$ transforms into a space derivative, the following action is invariant under N=2 super-transformation:

$$
\begin{aligned}
S &= \int d^2x \int d^4\theta \ \Phi^+\Phi \\
&= \int d^2x \left[\frac{1}{4}\partial^{+\mu}\phi^*\partial_{-\mu}\phi - \frac{i}{4}\partial^{-\mu}\xi^+{}_\mu\xi - \frac{i}{4}\partial^{+\mu}\xi^-{}_\mu\xi^* - \frac{1}{4}K^-{}_\mu K^{+\mu}\right] \\
&= \int d^2x \left[\frac{1}{4}\partial_\mu\phi_i\partial^\mu\phi^i - \frac{i}{2}\partial^\mu\xi_\mu\psi - \frac{i}{2}\epsilon^{\mu\nu}\partial_\mu\xi_\nu\tilde{\psi} - \frac{1}{4}K_\mu K^\mu\right],
\end{aligned} \tag{4.68}
$$

where in the last equality we use the following redefinitions of fields:

$$
\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), K_\mu = \frac{1}{\sqrt{2}}(K^+{}_\mu + K^-{}_\mu), \\
\xi &= \frac{1}{\sqrt{2}}(\psi - i\tilde{\psi}), \ \xi_\mu = \frac{1}{\sqrt{2}}(\xi^+{}_\mu + \xi^-{}_\mu).
\end{aligned} \tag{4.69}
$$

Here again, the set of fields $(\psi, \xi_\mu, \tilde{\psi})$ in (4.69) lead matter fermion via Dirac-Kähler fermion mechanism:

$$
-\int d^2x \ \left(\frac{i}{2}\partial^\mu\xi_\mu\psi + \frac{i}{2}\epsilon^{\mu\nu}\partial_\mu\xi_\nu\tilde{\psi}\right) = \int d^2x \ \mathrm{Tr}\left(\frac{i}{2}\overline{\xi}\gamma^\mu\partial_\mu\xi\right), \tag{4.70}
$$

where

$$
\xi_{\alpha\beta} = \frac{1}{2}\left(\mathbf{1}\psi + \gamma^\mu\xi_\mu + \gamma^5\tilde{\psi}\right)_{\alpha\beta}, \tag{4.71}
$$

and $\overline{\xi} = C\xi^T C = \xi^T$.

As we will see later, these types of chiral superfields will appear when the Yang-Mills type action will be introduced. And then the fermionic contents $(\xi, \xi^*, \xi^+{}_\mu, \xi^-{}_\mu)$ will lead to matter fermion via Dirac-Kähler fermion mechanism again.

## 4.4 Vector superfields and gauge transformation

Vector superfield is defined as the superfield satisfying the condition

$$V = V^*, \tag{4.72}$$

and can be expanded as follows:

$$\begin{aligned}
&V(x^\mu, \theta^+, \theta^{+\mu}, \theta^-, \theta^{-\mu}) \\
&\quad = L(x) + \frac{1}{2}\theta^{+\mu}\eta^-{}_\mu(x) - \frac{1}{2}\theta^{-\mu}\eta^+{}_\mu(x) + \theta^-\eta(x) - \theta^+\eta^*(x) \\
&\qquad + \frac{1}{2}\theta^-\theta^{+\mu}M^-{}_\mu(x) - \frac{1}{2}\theta^+\theta^{-\mu}M^+{}_\mu(x) \\
&\qquad + \frac{1}{2}\theta^+\theta^{+\mu}\omega^-{}_\mu(x) - \frac{1}{2}\theta^-\theta^{-\mu}\omega^+{}_\mu(x) + \theta^+\theta^- A(x) + \frac{1}{2}\theta^{+\mu}\theta^-{}_\mu B(x) \\
&\qquad + \frac{1}{2}\theta^+\theta^{+\mu}\theta^-\left(\lambda^-{}_\mu(x) + \frac{i}{2}\partial^-{}_\mu\eta(x)\right) - \frac{1}{2}\theta^+\theta^{-\mu}\theta^-\left(\lambda^+{}_\mu(x) - \frac{i}{2}\partial^+{}_\mu\eta^*(x)\right) \\
&\qquad + \frac{1}{2}\theta^-\theta^{+\rho}\theta^-{}_\rho\left(\lambda(x) + \frac{i}{4}\partial^+{}_\mu\eta^{-\mu}(x)\right) - \frac{1}{2}\theta^+\theta^{+\rho}\theta^-{}_\rho\left(\lambda^*(x) - \frac{i}{4}\partial^-{}_\mu\eta^{+\mu}(x)\right) \\
&\qquad + \frac{1}{2}\theta^+\theta^-\theta^{+\rho}\theta^-{}_\rho\left(D(x) - \frac{1}{4}\partial^2 L(x)\right),
\end{aligned} \tag{4.73}$$

where $L$, $A$, $B$, and $D$ must all be real.

We consider the following transformation:

$$V \to V' = V + \Lambda + \Lambda^+, \tag{4.74}$$

where $\Lambda^+ = \Lambda^*$ and thus $\Lambda + \Lambda^+$ satisfies the vector superspace condition (4.72). Here $\Lambda$ and $\Lambda^+$ are chiral superfields,

$$\begin{aligned}
\Lambda = {} & v(x) + \frac{1}{2}\theta^{+\mu}\zeta^-{}_\mu(x) + \theta^-\zeta(x) \\
& + \frac{i}{4}\theta^+\theta^{+\mu}\partial^-{}_\mu v(x) - \frac{i}{4}\theta^-\theta^{-\mu}\partial^+{}_\mu v(x) + \frac{1}{2}\theta^-\theta^{+\mu}N^-{}_\mu(x) \\
& + \frac{i}{8}\theta^-\theta^{+\rho}\theta^-{}_\rho\partial^{+\mu}\zeta^-{}_\mu(x) + \frac{i}{4}\theta^+\theta^{+\mu}\theta^-\partial^-{}_\mu\zeta(x) \\
& - \frac{1}{8}\theta^+\theta^-\theta^{+\mu}\theta^-{}_\mu\partial^2 v(x),
\end{aligned} \tag{4.75}$$

and

$$\begin{aligned}
\Lambda^+ = \Lambda^* = {} & v^*(x) - \frac{1}{2}\theta^{-\mu}\zeta^+{}_\mu(x) - \theta^+\zeta^*(x) \\
& - \frac{i}{4}\theta^+\theta^{+\mu}\partial^-{}_\mu v^*(x) + \frac{i}{4}\theta^-\theta^{-\mu}\partial^+{}_\mu v^*(x) - \frac{1}{2}\theta^-\theta^{+\mu}N^+{}_\mu(x) \\
& + \frac{i}{8}\theta^+\theta^{+\rho}\theta^-{}_\rho\partial^{-\mu}\zeta^+{}_\mu(x) + \frac{i}{4}\theta^+\theta^{-\mu}\theta^-\partial^+{}_\mu\zeta^*(x) \\
& - \frac{1}{8}\theta^+\theta^-\theta^{+\mu}\theta^-{}_\mu\partial^2 v^*(x),
\end{aligned} \tag{4.76}$$

where $D^+\Lambda = D^-{}_\mu\Lambda = 0$ and $D^-\Lambda^+ = D^+{}_\mu\Lambda^+ = 0$. Under this transformation, the component fields of $V$ transform as follows:

$$\begin{aligned}
L &\to L + v + v^*, \\
\eta^\pm{}_\mu &\to \eta^\pm{}_\mu + \zeta^\pm{}_\mu, \\
\eta &\to \eta + \zeta, \\
M^\pm{}_\mu &\to M^\pm{}_\mu + N^\pm{}_\mu, \\
\omega^\pm{}_\mu &\to \omega^\pm{}_\mu + \frac{i}{2}\partial^\pm{}_\mu(v - v^*), \\
A &\to A, \\
B &\to B, \\
\lambda^\pm{}_\mu &\to \lambda^\pm{}_\mu, \\
\lambda &\to \lambda, \\
D &\to D.
\end{aligned} \tag{4.77}$$

While $\omega_\mu = \frac{1}{\sqrt{2}}(\omega^+{}_\mu + \omega^-{}_\mu)$ transforms as:

$$\omega_\mu \to \omega_\mu + \partial_\mu \mathrm{Im}(v), \tag{4.78}$$

where we can identify $\omega_\mu$ and $\mathrm{Im}(v)$ as the gauge field and the gauge parameter, respectively. Then the transformation (4.74), or (4.77), can be regarded as the supersymmetric generalization of the gauge transformation.

We introduce the following superfields:

$$W = \frac{1}{2}D^-D^-{}_\mu D^{+\mu}V, \tag{4.79}$$

$$W_\mu = D^+{}_\mu D^- D^+ V. \tag{4.80}$$

These are chiral superfields:

$$\begin{aligned}
D^-W = 0,\ \ D^+{}_\mu W = 0, \\
D^-W_\mu = 0,\ \ D^+{}_\rho W_\mu = 0,
\end{aligned} \tag{4.81}$$

and they are gauge invariant under the gauge transformation (4.74),

$$W \to \frac{1}{2}D^+D^+{}_\mu D^{-\mu}(V + \Lambda + \Lambda^+) = W, \tag{4.82}$$

$$W_\mu \to D^-{}_\mu D^+ D^-(V + \Lambda + \Lambda^+) = W_\mu. \tag{4.83}$$

In the following subsection, we introduce the super and gauge invariant action described by the chiral super fields $(\Phi, W, W_\mu)$, anti-chiral super field $\Phi^+$ and the vector superfield $V$.

## 4.5 Twisted super Yang-Mills action

We introduce the following action:

$$S = \frac{1}{2}\int d^2x \int d\theta^{+\mu} d\theta^- W W_\mu + 4\int d^2x \int d^4\theta\ \Phi^+ e^{gV}\Phi, \tag{4.84}$$

where $g$ is a constant, $\Phi$ and $\Phi^+$ are chiral superfields (4.64) and (4.65), respectively. The gauge transformations of the chiral superfield are defined as follows:

$$\begin{aligned} \Phi &\longrightarrow \Phi' = e^{-g\Lambda}\Phi, \\ \Phi^+ &\longrightarrow \Phi'^+ = \Phi^+ e^{-g\Lambda^+}. \end{aligned} \tag{4.85}$$

Since $W$ and $W_\mu$ are chiral superfields, $\theta^{+\mu}\theta^-$ component of $WW_\mu$ transforms into a space derivative, and therefore the first term in (4.84) is super and gauge invariant. The second term is trivially super and gauge invariant.

By using the degrees of freedom of the gauge parameters except $\mathrm{Im}(v)$, we can take a special gauge, Wess-Zumino gauge, in which $L$, $\eta^\pm{}_\mu$, $\eta$, and $M^\pm{}_\mu$ are all set to be zero by adjusting the parameters; $v+v^*, \zeta^\pm{}_\mu, \zeta$, and $N^\pm{}_\mu$ respectively:

$$\begin{aligned} &V(x^\mu, \theta^+, \theta^{+\mu}, \theta^-, \theta^{-\mu}) \\ &= \frac{1}{2}\theta^+\theta^{+\mu}\omega^-{}_\mu - \frac{1}{2}\theta^-\theta^{-\mu}\omega^+{}_\mu + \theta^+\theta^- A + \frac{1}{2}\theta^{+\mu}\theta^-{}_\mu B \\ &\quad + \frac{1}{2}\theta^+\theta^{+\mu}\theta^-\lambda^-{}_\mu - \frac{1}{2}\theta^+\theta^{-\mu}\theta^-\lambda^+{}_\mu \\ &\quad + \frac{1}{2}\theta^-\theta^{+\rho}\theta^-{}_\rho\lambda - \frac{1}{2}\theta^+\theta^{+\rho}\theta^-{}_\rho\lambda^* \\ &\quad + \frac{1}{2}\theta^+\theta^-\theta^{+\rho}\theta^-{}_\rho D. \end{aligned} \tag{4.86}$$

Then $W$ and $W_\mu$ turn out as follows:

$$\begin{aligned} W(y^\mu, \theta^{+\mu}, \theta^-) &= \lambda^* - \theta^-\left(D - \frac{i}{4}\partial^{+\rho}\omega^-{}_\rho + \frac{i}{4}\partial^{-\rho}\omega^+{}_\rho\right) \\ &\quad + \frac{i}{2}\theta^{+\mu}\partial^-{}_\mu B - \frac{i}{2}\theta^-\theta^{+\mu}\partial^-{}_\mu\lambda, \end{aligned} \tag{4.87}$$

$$\begin{aligned} W_\mu(y^\mu, \theta^{+\mu}, \theta^-) &= -\lambda^+{}_\mu + i\theta^-\partial^+{}_\mu A \\ &\quad + \theta^+{}_\mu\left(D + \frac{i}{4}\partial^{+\rho}\omega^-{}_\rho - \frac{i}{4}\partial^{-\rho}\omega^+{}_\rho\right) + \frac{i}{2}\theta^+{}_\mu\theta^-\partial^{+\rho}\lambda^-{}_\rho. \end{aligned} \tag{4.88}$$

Once we take the Wess-Zumino gauge, the structure of supersymmetry breaks down. The action is, however, invariant under a new supersymmetry transformation combined with the gauge transformation; $\delta = \delta_{super} + \delta_{gauge}$. We can redefine this combined transformation as the new super transformation, where the new supersymmetry algebra closes up to the gauge transformation.

Now the action (4.84) in the Wess-Zumino gauge turns out as:

$$S = \int d^2x \bigg[ 2\left(\frac{i}{4}\partial^{+\mu}\omega^-{}_\mu - \frac{i}{4}\partial^{-\mu}\omega^+{}_\mu\right)^2 + i\partial^{-\mu}\lambda^+{}_\mu\lambda + i\partial^{+\mu}\lambda^-{}_\mu\lambda^* + \partial^+{}_\mu A\partial^{-\mu}B - 2D^2$$
$$- 2\partial^2\phi^*\phi - K^-{}_\mu K^{+\mu} - i\partial^{-\mu}\xi^+{}_\mu\xi - i\partial^{+\mu}\xi^-{}_\mu\xi^*$$
$$+ g\Big( -\frac{i}{2}\phi\omega^-{}_\mu\partial^{+\mu}\phi^* + \frac{i}{2}\phi^*\omega^-{}_\mu\partial^{+\mu}\phi - \frac{i}{2}\phi\omega^+{}_\mu\partial^{-\mu}\phi^* + \frac{i}{2}\phi^*\omega^+{}_\mu\partial^{-\mu}\phi$$
$$- \xi\omega^-{}_\mu\xi^{+\mu} + \xi^*\omega^+{}_\mu\xi^{-\mu} + A\xi^-{}_\mu\xi^{+\mu} - 2B\xi\xi^*$$
$$+ \lambda^{+\mu}\xi^-{}_\mu\phi^* - \lambda^{-\mu}\xi^+{}_\mu\phi + 2\lambda^*\xi\phi^* - 2\lambda\xi^*\phi + 2D\phi\phi^* \Big)$$
$$+ g^2\left(\omega^-{}_\mu\omega^{+\mu}\phi\phi^* + 2AB\phi\phi^*\right)\bigg]. \tag{4.89}$$

We redefine the component fields as:

$$\begin{aligned}
\phi &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2), \omega_\mu = \frac{1}{\sqrt{2}}(\omega^+{}_\mu + \omega^-{}_\mu), \\
\lambda &= \frac{1}{\sqrt{2}}(\rho - i\tilde{\rho}), \lambda_\mu = \frac{1}{\sqrt{2}}(\lambda^+{}_\mu + \lambda^-{}_\mu), \\
\xi &= \frac{1}{\sqrt{2}}(C_1 - iC_2), K_\mu = \frac{1}{\sqrt{2}}(K^+{}_\mu + K^-{}_\mu), \\
\chi_{\mu 1} &= \frac{1}{\sqrt{2}}(\xi^+{}_\mu + \xi^-{}_\mu), \chi_{\mu 2} = \frac{1}{\sqrt{2}}\epsilon_{\mu\nu}(\xi^{+\nu} + \xi^{-\nu}),
\end{aligned} \tag{4.90}$$

where $\chi_{\mu i}$ is the anti-selfdual field which obeys the condition $\chi_{\mu i} = -\epsilon_{\mu\nu}\epsilon_{ij}\chi^{\nu j}$.

Then the action can be rewritten as follows:

$$\begin{aligned}
S = \int d^2x \bigg[ &\frac{1}{2}F^{\mu\nu}F_{\mu\nu} + 2i\partial^\mu\lambda_\mu\rho + 2i\epsilon^{\mu\nu}\partial_\mu\lambda_\nu\tilde{\rho} - 2A\partial^2 B \\
&- 2D^2 + gD\phi_i\phi^i - K_\mu K^\mu \\
&+ (D_\mu\phi)_i(D^\mu\phi)^i + 2i\chi_{\mu i}(D_\mu C)^i \\
&+ \sqrt{2}ig\lambda^\mu\epsilon^{ij}\chi_{\mu i}\phi_j + \sqrt{2}ig(\rho\epsilon^{ij}C_i\phi_j + \tilde{\rho}C_i\phi^i) \\
&+ \frac{i}{2}gA\epsilon^{ij}\chi_{\mu i}\chi^\mu{}_j - igB\epsilon^{ij}C_iC_j + g^2AB\phi_i\phi^i \bigg],
\end{aligned} \tag{4.91}$$

where $F_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu$, and $(D_\mu\phi)_i = \partial_\mu\phi_i - \frac{1}{\sqrt{2}}g\omega_\mu\epsilon_{ij}\phi^j$. This action is the off-shell N=2 supersymmetry invariant extended version of the action (3.33) obtained from partially gauge fixing of the generalized topological Yang-Mills action with the instanton conditions. Indeed, integrating $D$ and $K_\mu$, the action (4.91) turns out to

be equivalent to the action (3.33):

$$
\begin{aligned}
S = \int d^2x \Big[ & \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + 2i\partial^\mu\lambda_\mu\rho + 2i\epsilon^{\mu\nu}\partial_\mu\lambda_\nu\tilde{\rho} - 2A\partial^2 B \\
& + |\phi|^4 + (D_\mu\phi)_i(D^\mu\phi)^i + 2i\chi_{\mu i}(D_\mu C)^i \\
& + 4i\lambda^\mu\epsilon^{ij}\chi_{\mu i}\phi_j + 4i(\rho\epsilon^{ij}C_i\phi_j + \tilde{\rho}C_i\phi^i) \\
& + \sqrt{2}iA\epsilon^{ij}\chi_{\mu i}\chi^\mu{}_j - 2\sqrt{2}iB\epsilon^{ij}C_iC_j + 8AB|\phi|^2 \Big],
\end{aligned}
\tag{4.92}
$$

where we set $g = 2\sqrt{2}$. Here the sets $(\phi_i, C_i, \chi_{\mu i})$ and $(A, B, \omega_\mu, \rho, \lambda_\mu, \tilde{\rho})$ are the N=2 chiral multiplet and the N=2 vector multiplet, respectively, and their gauge transformations are given by the transformations (4.85) and (4.74) in the Wess-Zumino gauge:

$$
\begin{aligned}
\delta_{gauge}(\phi_i, C_i, \chi_{\mu i}) &= 2\mathrm{Im}(v)\epsilon_{ij}(\phi^j, C^j, \chi_\mu{}^j), \\
\delta_{gauge}\omega_\mu &= \partial_\mu v, \\
\delta_{gauge}(A, B, \rho, \lambda_\mu, \tilde{\rho}) &= 0,
\end{aligned}
\tag{4.93}
$$

where $\mathrm{Im}(v)$ is the gauge parameter. These transformation of the fields is the $SO(2)$ gauge transformation (3.34).

# 5 Twisting as Dirac-Kähler fermion mechanism

We have observed in the quantization of the generalized topological Yang-Mills theory that a linear combination of the ghosts of shift symmetry of the topoligical nature of generalized Yang-Mills; $\tilde{C}_\mu$ and $\tilde{C}_a$, the anti-ghosts of instanton gauge fixing; $\chi_{\mu a}$ and $\lambda$, and the Lagrange multiplier of ghost symmetry; $\rho$, turn into matter fermions via the twisting procedure which we identify as Dirac-Kähler fermion mechanism[12]. In other words the vector or tensor suffices of ghost fields and Lagrange multiplier can be transformed into spinor suffices as in the eqs. (3.36) and (3.37):

$$\psi_{\alpha i} = \tfrac{1}{2}(\rho + \gamma^\mu\tilde{C}_\mu - \gamma^5\lambda)_{\alpha i}, \tag{5.1}$$

$$\chi_{\alpha i} = \tfrac{1}{2}(-\tilde{C}_1 + \gamma^\mu\chi_{\mu 1} - \gamma^5\tilde{C}_2)_{\alpha i}. \tag{5.2}$$

It works exactly in the same way for the fields of (4.49) and (4.54).

$$\xi_{\alpha i} = \frac{1}{2}\left(\mathbf{1}\chi + \gamma^\mu\phi_\mu + \gamma^5\tilde{\chi}\right)_{\alpha i}, \tag{5.3}$$

$$\psi_{\alpha i} = \frac{1}{2}\left(\mathbf{1}\psi + \gamma^\mu\psi_\mu + \gamma^5\tilde{\psi}\right)_{\alpha i}. \tag{5.4}$$

The super charges of N=2 supersymmetry and the twisted N=2 super charges have the following relation:

$$Q_{\alpha i} = \left(\mathbf{1}Q + \gamma^\mu Q_\mu + \gamma^5\tilde{Q}\right)_{\alpha i}. \tag{5.5}$$

This leads to the following algebra including the angular momentum generator $J$ and R-symmetry generator $R$:

$$
\begin{aligned}
[J, Q_{\alpha i}] &= \tfrac{i}{2}(\gamma^5)_\alpha{}^\beta Q_{\beta i}, \\
[R, Q_{\alpha i}] &= \tfrac{i}{2}(\gamma^5)_i{}^j Q_{\alpha j}
\end{aligned}
\tag{5.6}
$$

and the algebra including the angular momentum generator of the twisted space $J'$ and $R$:

$$
\begin{aligned}
&[J', Q] = [J', \tilde{Q}] = 0,\ [J', Q_\mu] = i\epsilon_{\mu\nu}Q^\nu, \\
&[R, Q] = \tfrac{i}{2}\tilde{Q},\ [R, Q_\mu] = \tfrac{i}{2}\epsilon_{\mu\nu}Q^\nu,\ [R, \tilde{Q}] = -\tfrac{i}{2}Q.
\end{aligned}
\tag{5.7}
$$

Those generators have the following relation:

$$
J' = J + R. \tag{5.8}
$$

As we can see from the above relations, the angular momentum generator $J$ generates the Lorentz rotation of the spinor suffix and thus the spinor fields $\psi_{\alpha i}, \chi_{\alpha i}$ and $\xi_{\alpha i}, \psi_{\alpha i}$ have half integer spin while the R-symmetry generator $R$ rotates the "flavor" suffix $i$ of those fermion fields. It is, however, interesting to recognize that the twisted angular momentum generator $J'$ rotates the scalar, vector and tensor suffix of twisted super charges. In other words the fermion fields with scalar, vector and tensor suffixes transform as integer spin fermion fields. Thus the R-symmetry generator $R$ plays the role of shifting the integer spin of ghost fermions into the half integer spin of the matter fermions.

This situation can be more clearly seen by the following concrete example. The Lorentz transformation on the Dirac-Kähler field $\psi$ of (5.1) induced by $J'$ is given by

$$
\begin{aligned}
\delta_{J'}\psi &= \frac{1}{2}\Big(i\epsilon_\mu{}^\nu \tilde{C}_\nu \gamma^\mu\Big) \\
&= i\frac{1}{2}[\gamma_5, \psi].
\end{aligned}
\tag{5.9}
$$

On the other hand the Lorentz transformation induced by $J = J' - R$ is

$$
\begin{aligned}
\delta_J\psi &= \delta_{J'}\psi - \delta_R\psi \\
&= i\frac{1}{2}\gamma_5\psi,
\end{aligned}
\tag{5.10}
$$

which precisely coincides with the Lorentz transformation of spinor field since the R-symmetry rotation $\delta_R\psi$ is the rotation of the "flavor" suffix $i$ of the matter fermions. This shows that Dirac-Kähler fermion mechanism is essentially related to the twisting procedure with $N = 2$ supersymmetry.

## 6 Conclusions and Discussions

We have explicitly shown that the BRST charge; $s$, the vector supersymmetry charge; $s_\mu$, and pseudo scalar supersymmetry charge; $\tilde{s}$, constitute the twisted super charges of D=N=2 super symmetry and are related to the N=2 supersymmetry charges; $Q$, in the following simple relation:

$$Q = \left(\mathbf{1}s + \gamma^\mu s_\mu + \gamma^5 \tilde{s}\right), \tag{6.1}$$

which has the same algebraic structure as the Dirac-Kähler fermion formulation. R-symmetry of N=2 super algebra is the "flavor" symmetry of the Dirac-Kähler field.

Based on this supersymmetry algebra we have proposed the twisted D=N=2 superspace formalism. We have defined the twisted superspace in terms of explicitly defined differential operators. We first have constructed models with the chiral superfield set $(\Psi, \overline{\Psi})$:

$$S = (-i)^{\epsilon_\Psi} \int d^2x d^4\theta \ \mathrm{Tr}\ (\overline{\Psi}\Psi), \tag{6.2}$$

where $\Psi$ and $\overline{\Psi}$ are chiral and anti-chiral superfields for Abelian case while the chiral condition is broken for the non-Abelian case. We have explicitly shown that a fermionic bilinear form of twisted N=2 chiral and anti-chiral superfields is equivalent to the quantized version of BF theory with the Landau type gauge fixing while a bosonic bilinear form leads to the N=2 Wess-Zumino action after nontrivial change of field variables. In showing these equivalences we have found that the ghost fields turn into matter fermion by the twisting mechanism. We claim that this twisting mechanism is nothing but the Dirac-Kähler fermion formulation, which is the essence of the twisted superspace formulation.

Secondly, we have constructed the super and gauge invariant action with the chiral superfields $(\Phi, \Phi^+)$ and the vector superfield $V$,

$$S = \int d^2x \int d\theta^{+\mu} d\theta^- W W_\mu + 4 \int d^2x \int d^4\theta \ \Phi^+ e^{gV} \Phi. \tag{6.3}$$

We have shown that the action in the Wess-Zumino gauge turns out to be off-shell twisted N=2 supersymmetric action which leads to the partially gauge fixed action with the instanton conditions of the generalized topological Yang-Mills theory after integrating auxiliary fields. The close relations between the twisting mechanism and Dirac-Kähler fermion mechanism can be more explicitly seen in this example. The role of the R-symmetry and the change of the spin structre from ghost related fermions to matter fermions in the Dirac-Kähler fermion mechanism are explicitly shown.

The Dirac-Kähler fermion is formulated by introducing all the degrees of inhomogeneous differential forms which are then transformed into Dirac-Kähler fermion fields possessing the spinor and "flavor" suffixes. It is interesting to recognize that

the generalized gauge theory possesses all the degrees of differential forms as gauge parameters which then turn into ghost fields after the quantization of the generalized gauge fields[52, 53]. We expect that the quantized generalized gauge theory may lead to extended supersymmetry invariant actions even in other dimensions than two dimensions.

As we can see from the formulation, the generalization of the twisted superspace formalism into four dimensions is very important proplem to pursue since we expect to obtain off-shell N=4 supersymmetric actions in four dimensions. There are already some investigations on D=N=4 formulation of the topological field theories[7, 59, 60, 61]. Along the line of the formulation of this paper we can indeed generalize it into four dimensions, however, it is not easy to find the known actions having N=4 supersymmetry[54].

Another important issue to recognize is that Dirac-Kähler fermion is a curved spacetime version of Kogut-Susskind fermion [55, 56] or staggered fermion [57] and thus a natural framework of the lattice fermion formulation. It was shown in the recent paper[58] that the Dirack-Kähler fermion formulation can be defined in unambiguous way in terms of Clifford product by introducing noncommutative differential form on the lattice. Here the notorious problem of the difficulty of the Leipnitz rule on the lattice is avoided by paying the price of introducing noncommutativity. We would like to show that the supersymmetry on the lattice with the noncommutativity will be realized by using the formulation of the twisted superspace formalism of this article.

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# Appendix

# A. Notations

Throughout this paper, we consider the two-dimensional Euclidean space-time, where the $\gamma$- matrices satisfy

$$\begin{aligned} &\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}\mathbf{1}, \\ &\{\gamma^5, \gamma^\mu\} = 0, \\ &C\gamma^\mu C^{-1} = \gamma^{\mu T}, \end{aligned} \tag{A.1}$$

where $\gamma^5 = \frac{1}{2}\epsilon_{\mu\nu}\gamma^\mu\gamma^\nu = \gamma^1\gamma^2$ , $\epsilon_{12} = -\epsilon_{21} = \epsilon^{12} = -\epsilon^{21} = 1$, and $C$ is charge conjugation matrix which can be taken as $C = \mathbf{1}$ in the representation $\gamma^\mu = (\sigma^3, \sigma^1)$.

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